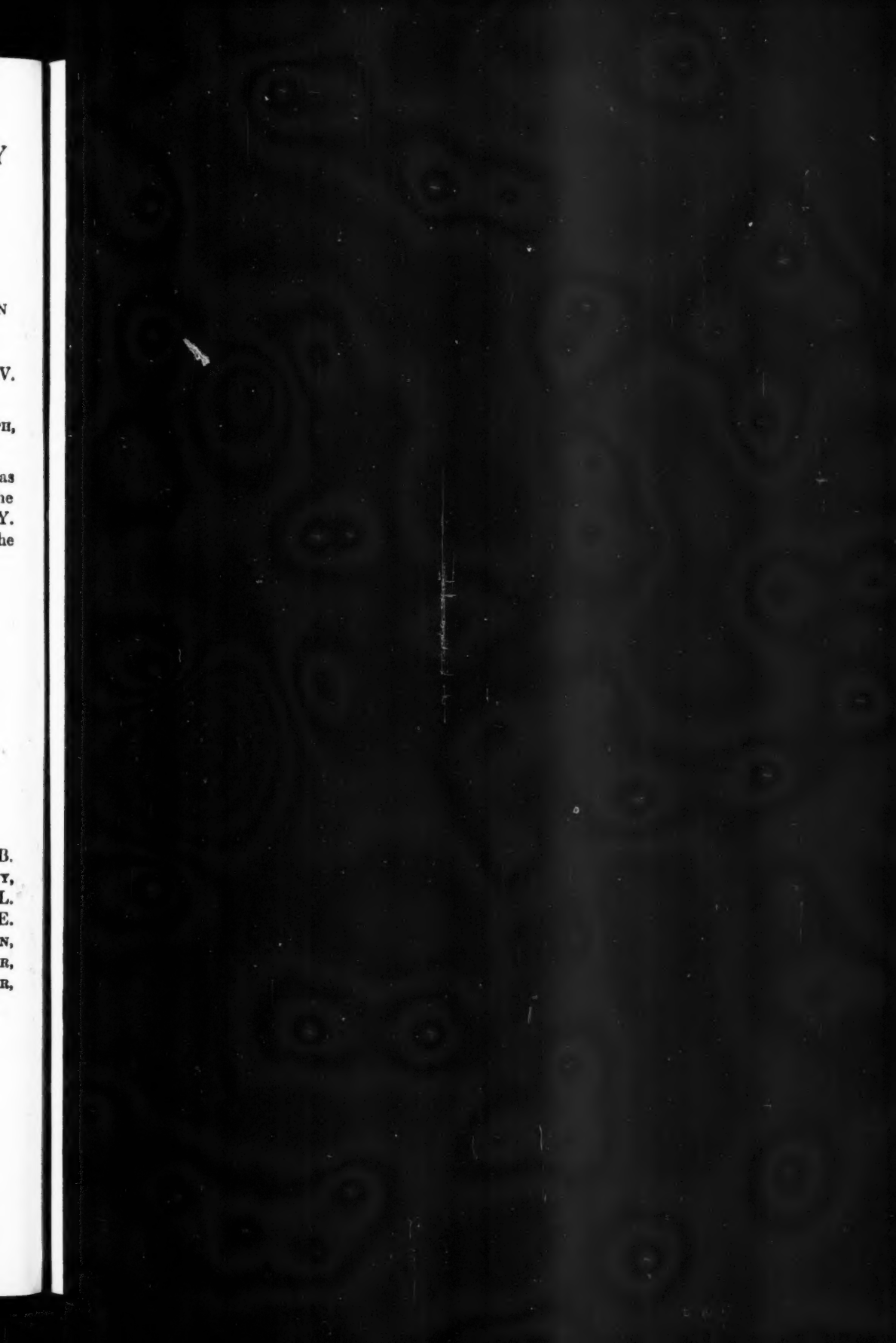

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THE FUTURE OF MATHEMATICS*

ANDRÉ WEIL

"At one time," says Poincaré in his Rome conference on the future of mathematics, "there were prophets of misfortune; they reiterated that all the problems had been solved, that after them there would be nothing but gleanings left. . . . " "But," he added, "the pessimists have always been compelled to retreat . . . so that I believe there are none left to-day."

Our faith in progress, our belief in the future of our civilization are no longer as strong; they have been too rudely shaken by brutal shocks. To us, it hardly seems legitimate to "extrapolate" from the past and present to the future, as Poincaré did not hesitate to do. If the mathematician is asked to express himself as to the future of his science, he has a right to raise the preliminary question: what kind of future is mankind preparing for itself? Are our modes of thought, fruits of the sustained efforts of the last four or five millennia, anything more than a vanishing flash? If, unwilling to stumble into metaphysics, one should prefer to remain on the hardly more solid ground of history, the same questions reappear, although in different guise: are we witnessing the beginning of a new eclipse of civilization? Rather than to abandon ourselves to the selfish joys of creative work, is it not our duty to put the essential elements of our culture in order, for the mere purpose of preserving it, so that at the dawn of a new Renaissance, our descendants may one day find them intact?

These questions are not purely rhetorical; upon each man's answer, or rather (for such questions do not have answers), upon the attitude which he takes in front of them, depends in large measure the trend of his intellectual efforts. It was necessary, before writing about the future of mathematics, to formulate these questions, just as the faithful cleansed themselves before consulting the oracle. Let us now interrogate destiny.

Mathematics, as we know it, appears to us as one of the necessary forms of our thought. The archaeologist and the historian have shown us civilizations from which mathematics were absent. It is indeed doubtful whether they would ever have become more than a technique, at the service of technologies, if it had not been for the Greeks; and it is possible that, under our very eyes, a type of human society is being evolved in which they will be nothing but that. But for us, whose shoulders sag under the weight of the heritage of Greek thought and who walk in the paths traced out by the heroes of the Renaissance, a civilization without mathematics is unthinkable. Like the parallel postulate, the postulate that mathematics will survive has been stripped of its "evidence"; but, while the former is no longer necessary, we would not be able to get on without the latter.

The clinical student of ideas who limits his prognosis to the immediate fu-

* Authorized translation by Arnold Dresden of the article entitled L'avenir des mathématiques in the volume Les grands courants de la pensée mathématique, edited by F. Le Lionnais. Cahiers du Sud, Marseille, 1948.

ture, and does not risk long-range prophecies, certainly observes more than one favorable symptom in contemporary mathematics. To begin with, while some sciences, conferring, as they now do, an almost unlimited power upon a ruthless possessor of their results, tend to become caste monopolies, treasures jealously guarded under a seal of secrecy which must of necessity become fatal to any genuine scientific activity, the real mathematician does not seem to be exposed to the temptations of power nor to the straight-jacket of state secrecy. "Mathematics" said G. H. Hardy in substance in a famous inaugural lecture, "is a useless science. By this I mean that it can contribute directly neither to the exploitation of our fellowmen, nor to their extermination."

It is certain that few men of our times are as completely free as the mathematician in the exercise of their intellectual activity. Even if some State ideologies sometimes attack his person, they have never yet presumed to judge his theorems. Every time that so-called mathematicians, to please the powers that be, have tried to subject their colleagues to the yoke of some orthodoxy, their only reward has been contempt. Let others besiege the offices of the mighty in the hope of getting the expensive apparatus, without which no Nobel prize comes within reach. Pencil and paper is all the mathematician needs; he can even sometimes get along without these. Neither are there Nobel prizes to tempt him away from slowly maturing work, towards a brilliant but ephemeral result. Mathematics is taught the world over, well here, badly there; the exiled mathematician—and who among us can to-day feel free from the danger of exile—can find everywhere the modest livelihood which allows him to pursue his work to some extent. Even in gaol one can do good mathematics if one's courage fail him not.

To these "objective conditions," or rather, as the physician would say, to these external symptoms, must be added others revealed by a more penetrating clinical examination. In recent times mathematics has demonstrated its vitality by passing through one of these periods of growing pains, to which it has been accustomed for a long time, and which are designated by the strange name of "foundation crises." It has come through it, not only without damage, but with great gain. Whenever wide domains have been added to the field of mathematical reasoning, it is necessary to inquire what techniques are allowed in the exploration of the new territory. One wants certain objects to have certain properties, one wants certain modes of reasoning to be admissible and one behaves as if they were. But the pioneer who proceeds in this way knows very well that some day the police will come to put an end to the disorder and to bring everything under the control of the general law. Thus, when the Greeks defined the ratio of two magnitudes for the first time with enough precision to raise the problem of the existence of incommensurable magnitudes, they seem to have believed and to have wanted all ratios to be rational and to have based the first sketch of their geometrical reasonings on this provisional hypothesis; some of the greatest advances in Greek mathematics are connected with the discovery of their initial error at this point. In the same way, at the beginning of the era

of the theory of functions and of the infinitesimal calculus, one wished every analytical expression to define a function, and every function to have a derivative; we know to-day that these requirements were incompatible. The last crisis, which grew out of the sophistries, for which the "naive" theory of sets opened a way in its early stages, has led for us to a no less happy result, which can now be considered as permanently established. We have learned to trace our entire science back to a single source, constituted by a few signs and by a few rules for their use; this is unquestionably an unassailable stronghold, inside which we could scarcely confine ourselves without risk of famine, but to which we are always free to retire in case of uncertainty or of external danger. Only a few backward spirits still maintain the position that the mathematician must forever draw on his "intuition" for new, alogical or "prelogical" elements of reasoning. If certain branches of mathematics have not yet been axiomatized, *i.e.*, reduced to a form of exposition in which all terms are defined, and all axioms made explicit, in terms of the basic notions of set theory, this is simply because there has not yet been the time to do it. It is of course possible that some day our successors will want to introduce into set-theory modes of reasoning which we do not admit. It is even possible that the germ of a contradiction, which we do not perceive to-day, may later be discovered in the modes of reasoning we now use, although the work of the modern logicians makes this very unlikely. A general revision will then become necessary; one can feel certain even now that this will not affect the essential elements of our science.

But, if logic is the hygiene of the mathematician, it is not his source of food; the great problems furnish the daily bread on which he thrives. "A branch of science is full of life," said Hilbert, "as long as it offers an abundance of problems; a lack of problems is a sign of death." They are certainly not lacking in our mathematics; and the present time might not be ill chosen for drawing up a list, as Hilbert did in the famous lecture from which we have just quoted. Even among those of Hilbert, there are still several which stand out as distant, although not inaccessible, goals which will continue to suggest research for perhaps more than a generation; an example is furnished by his fifth problem, on Lie groups. The Riemann hypothesis, after the attempts to prove it by function-theoretic methods had been given up, appears to-day in a new light, which shows it to be closely connected with the conjecture of Artin on the L-functions, thus making these two problems two aspects of the same arithmetico-algebraic question, in which the simultaneous study of all the cyclotomic extensions of a given number field will undoubtedly play a decisive role. Gaussian arithmetic was centered around the law of quadratic reciprocity; we know now that this law is only a first example, we might better say the pattern of, the laws of "class fields," which control the abelian extensions of algebraic number-fields; we know how to formulate these laws so as to make them look like a coherent set. But, pleasant as this façade may be to the eye, we do not know whether it might not hide deeper lying symmetries. The automorphisms induced in the class groups by the automorphisms of the field, the properties of the norm-residues in the

non-cyclic cases, the passage to the limit (inductive or projective) when the base field is replaced by extensions, for example, cyclotomic extensions, of indefinitely increasing degree, all these are questions on which our ignorance is almost complete and in whose study the key to the Riemann hypothesis is perhaps to be found. Closely connected with these questions is the study of Artin's conductor and, in particular, in the local case, the search for the representation, whose trace can be expressed by means of simple characters with coefficients equal to the exponents of their conductors. These are some of the directions which can and must be followed up in order to penetrate the mystery of non-abelian extensions; it is not impossible that we are here close to principles of extraordinary fertility and that, once the first decisive step on this road will have been taken, we shall gain access to vast domains whose existence is hardly suspected. For, however wide our generalizations of Gauss' results may be, we can hardly claim to have as yet really moved beyond them.

Even in the realm of abelian extensions, we have not made any progress towards the generalization of the theorems of "Kronecker's youth dream," the generation of class fields, whose existence is known, by means of values of analytic functions. While it has been possible, without serious difficulties, to complete Kronecker's unfinished work and to obtain the solution of this problem, in the case of imaginary quadratic fields, by means of complex multiplication, the key to the general problem, considered by Hilbert as one of the most important of modern mathematics, still escapes us, in spite of the conjectures of Hilbert himself and the efforts of his pupils. Must we look for it perhaps in the new automorphic functions of Siegel, in his modular functions of several variables? Or can the theory of the endomorphisms of abelian varieties, which has now made considerable progress, be of some help here? It is too early to risk acceptable conjectures on these questions; but their closer examination is bound to produce interesting results, even though they should be negative in character.

The foregoing discussion shows clearly not only the vitality of modern arithmetic, but also the close ties which connect it, to-day as in the days of Euler and the days of Jacobi, with the most deep-lying parts of the theory of groups and of the theory of functions. This essential unity, which appears in so many and in such diverse ways, is also found in many other places. The introduction by Hermite of continuous variables into the theory of numbers has led to the systematic study of discontinuous groups of arithmetical nature by means of the continuous groups in which they can be imbedded, of the symmetric Riemannian spaces associated with these groups, of the differential and topological properties of their fundamental domains (or rather, in modern terminology, of their quotient spaces), and of the automorphic functions which belong to them. The work of Siegel, continuing the great tradition of Dirichlet, of Hermite and of Minkowski, has opened entirely new paths here. On the one hand, we connect with Fermat, Lagrange and Gauss, the representation of numbers by forms, and the genera of quadratic forms. At the same time, we begin to see in outline the fertile principle, according to which the global aspect of an arithmetical problem

can, under certain circumstances, be reconstructed from its local aspects. For instance, in the work of Siegel we see repeatedly that the number of solutions of some arithmetical problem in the field of rational numbers is expressed by means of numbers defined by the corresponding local problems, density of solutions in the real field and in the p-adic fields for all values of the prime p. This is a principle, analogous to Cauchy's theorem for the Riemann surface of an algebraic curve, with which one may connect also the famous "singular series" which appear in the application of the method of Hardy-Littlewood to problems in the analytic theory of numbers. Is it possible to formulate this principle in a general statement, which would allow us to obtain at one stroke all results of this character, just as the discovery of Cauchy's theorem made it possible to calculate by a single method a number of integrals and of series which were formerly treated by special distinct processes? It looks as if this were not yet a problem for the immediate future; so much the more reason to prepare for its solution by the study of well-chosen particular cases. It may be that this same principle will one day reveal the deep reason for the existence of Eulerian products, of which the extreme importance for the theory of numbers and the theory of functions has only become clear through the work of Hecke. Here we deal with the classes of quadratic forms, and not merely, as in the work of Siegel, with their genera; at the same time, we find ourselves at the core of the theory of modular functions, which has been infused with new life by these studies, and of the theory of theta functions. This domain is still so full of mystery, the questions which it raises so numerous and fascinating, that it would be premature to try to arrange them in order of importance.

At the same time, Siegel has taught us to construct discontinuous groups and automorphic functions by arithmetical methods; in this field the theory of functions, by its own efforts, had been unable to move forward since Poincaré. Indeed it is very likely, that, just as in the case of functions of a single variable, the thorough study of special functions of several complex variables will have to prepare the ground for an attack on the general theory. In the work of Siegel, the local and global geometrical study of fundamental domains, in effect of manifolds with a complex-analytic structure, tends to occupy a dominating role. Along this road, connection is made with the immense work accomplished by E. Cartan and its various extensions; at the same time one gets into the center of modern topology, the theory of fibre-spaces; and the invariants of Sitefel-Whitney appear, along with their generalizations. The intimate connections between these two domains had been suspected for some time, but their actual merging was made possible only through the recent discoveries of Chern, stimulated in their turn, at least in part, by considerations of algebraic geometry. Indeed, algebraic varieties, at least varieties without singularities in the complex field, are nothing but a special, and particularly interesting, class of manifolds with a complex-analytic structure; more precisely, they are manifolds on which, at least in all known cases, one can define one of these remarkable Hermitian metrics which were introduced by Kähler in connection with functions

of several complex variables, and of which results of S. Bergmann, not yet fully clarified, furnish other examples. By a systematic, although not explicit use of these metrics, Hodge has recently obtained the first existence theorems for this type of manifold, generalizing the classical results of Riemann. While it may be too much to hope that such methods may one day lead to the uniformization of algebraic varieties (which, contrary to what happens in the case of curves, can not be done in general by means of unramified functions), there is little doubt that they can be extended to integrals of the third kind. The analogous generalization of the methods of Hodge to differential forms with singularities in the real domain raises still more important problems. It appears to be connected, on the one hand, with local properties of the equations of elliptic type which harmonic forms satisfy; on the other hand, it seems to be linked with an extension of de Rham's theory which would make it possible to obtain the homologic torsion of a manifold by means of differential forms with singularities. De Rham's results have, as a matter of fact, definitely clarified a certain aspect of the relation between homology groups and multiple integrals, and this accounts for the fundamental role they play in the work of Hodge and of Chern; but until now they have only made the homology groups with real coefficients accessible to differential methods; moreover, the striking and fertile analogy between chains and differential forms, which is expressed in these results, remains a mere heuristic principle, until we succeed in finding a common basis for these two concepts. To convert this principle into a method of proof has thus far succeeded only in a few special cases, for example, in some of the beautiful papers with which Ahlfors has in recent years given fresh life to the theory of analytic functions.

But, while algebraic geometry, as we have just seen, receives a fresh stimulus from the most recent developments in topology and in differential geometry, this field does not lack purely algebraic problems; and, thanks to the methods of modern algebra, our understanding of them no longer need depend upon flashes of intuition of a few privileged mortals. At present, the theory of surfaces, brilliantly but too rapidly developed by the Italian school, must yield place to a general theory of algebraic varieties, freed from restrictive assumptions as to the nature of the base field and as to the absence of singularities. The structure of the groups of divisor classes, with respect to the different known concepts of equivalence (linear, continuous, numerical), the study of unramified extensions of a field of algebraic functions, both abelian and non-abelian, these are the questions that call for solution first of all. Thanks to the results obtained, or at least made plausible, by the Italian geometers, we can more or less guess the answers; and their solution, perhaps already within our reach, must open the road for important advances. On the other hand, the study of algebraic geometry over various special fields of constants is still gropingly taking its first steps. In view of the fact that algebraic geometry over the complex field, studied for almost a century, has arrived by its own methods (topological and transcendental) at well-known important results, it is probable that other fields,

finite fields, p-adic fields, fields of algebraic numbers, deserve to be studied, each by itself, by methods suited to their purpose. From this point of view, geometry over a finite field appears somewhat like a turntable, from which one may at will direct one's further progress either towards algebraic geometry proper, with the powerful tools already at its disposal, or towards the theory of numbers; it is precisely in that way that we are beginning to get a better insight into the nature of the Zeta-function and into the true nature of the Riemann hypothesis. In the same way, before undertaking the determination of the extensions of a field of algebraic numbers by means of their local properties, it might be indicated to solve the analogous problem, already difficult enough, concerning algebraic functions of one variable over a finite field, *i.e.*, to extend Riemann's existence theorem to such functions. To mention merely a particular case, one might ask whether the modular group, whose structure determines the fields of functions of a complex variable with only three points of ramification, plays the same role, at least with respect to the extensions of degree prime to the characteristic, when the field of constants is finite. It is not impossible that all questions of this kind can be treated by a uniform method, which would make it possble to deduce, from a result established (for instance by topological methods) for characteristic 0, the corresponding result for characteristic p; the discovery of such a principle would constitute an advance of the greatest importance. Of the same character, but still more difficult, are the problems arising in the modern study of finite groups. Is the theory of finite simple groups an analogue of the theory of simple Lie groups? It would probably be premature to make a frontal attack on this question at the present stage; by means of indirect procedures, in particular the study of p-groups, some progress has been made in this direction in recent years. As in many other questions of algebra and of the theory of numbers, so also here a new element has recently been introduced by the definition of the homology groups of an abstract group. The discovery of this concept, which generalizes the fruitful concepts of character and factor-set, is due to Eilenberg and MacLane, who introduced it in connection with H. Hopf's studies in pure combinatorial topology; it will have to be subjected for some time to systematic study, before its scope and possibilities of application can be estimated.

While arithmetic, in the widest sense, is for its devotees always the queen of mathematics, and while for that reason we have allowed ourselves to dwell on it with predilection, this is not to say that other branches of mathematics do not offer as many problems worthy of sustained attention. The work of a Cartan alone contains enough material to keep busy several generations of geometers. His general theory of systems in involution has not been carried to its conclusion by its author, who appears not to have been able to overcome all the difficulties of algebraic character which it involves. Concerning the theory of "infinite Lie groups," undoubtedly very important, but for us very obscure, we know nothing beyond what is found in the memoirs of Cartan, a first exploration into an almost impenetrable jungle; this jungle threatens to overgrow the

paths which already have been marked out, if the indispensable task of clearing is not undertaken very soon. The modern theory of Lie groups proper, studied by a combination of Cartan's methods with those of modern topology, is far from complete; even in the theory of semi-simple groups and in the theory of the symmetric Riemannian spaces associated with them, a good many results are attainable only by *a posteriori* verification, making use of our knowledge (also due to Cartan) of all simple groups. But, as has already been suggested, it is principally in the topological theory of fibre spaces, in the theorems of de Rham and in the notion of homotopy group, that we now find the tools best suited to the global study of the generalized geometries of Cartan. To give but a single example, the classical Gauss-Bonnet formula, until recently the only result in which a topological invariant was expressed by means of the integral of an invariant differential form, appears to us now as but the first term of a whole sequence of formulas, to which Chern's methods give access and whose systematic study has barely been started.

But, even though involutory systems should, in principle, enable us to obtain everything which can be reduced to the local problem of Cauchy-Kowalewski in the theory of partial differential equations, this is merely one aspect of the existence problem for solutions of these equations; and, from several points of view, it is not its most interesting aspect. Beyond this, we find important results concerning equations of very special types, chiefly elliptic and hyperbolic, some of which are of very recent date; but, although the study of these types, to which our predecessors were led by mathematical physics more than a century ago, is far from complete, it will not do to stop indefinitely at this point. The system which is satisfied by the real part of an analytic function of several complex variables does not belong to any of these simple types; however, function-theory has taught us, for example, that the most general singularities which they can have are, in a sense which is still not easy to specify, made up out of elementary singularities which are characteristic varieties; at any rate, one can interpret the theorems of Hartogs and of E. E. Levi in this manner. In this form, they present an obvious analogy to known results concerning hyperbolic equations; it is this analogy which suggests that we look for the germ of a general theory in a fuller development of the concepts of characteristic variety and of elementary solution. On the other hand, in the work of Delsarte, and in that of S. Bergmann and of his pupils, we find the first examples of the transformation of differential equations by means of integral or of integro-differential operators. It looks as if we have here the germ of entirely new developments and of a classification of systems of partial differential equations, which falls entirely outside the framework of classical methods. In particular, as was shown by Delsarte, the series of orthogonal functions to which elliptic problems lead naturally, are found to be transformed into series of much more general types; some isolated examples of these are found in classical analysis, but their general study presents problems of the greatest interest. The mathematician can no longer be satisfied here with Hilbert space, with which he has become as familiar as with

the Taylor series or the Lebesgue integral; must he look for the most appropriate tool in the theory of Banach spaces, or must he have recourse to more general spaces? It must be admitted that Banach spaces, interesting and useful as they have already proved to be, have not yet brought about the revolution in analysis which some people expected of them; but it would be a counsel of despair to abandon their study now before the various possibilities of application have been more fully explored. However it is possible that they are both too general to be suited to as exact a theory as that of Hilbert spaces, and too special to lend themselves to the study of the most significant operators. They do not include, for instance, the space of indefinitely differentiable functions; and it is only in that space that the operators of L. Schwartz can be defined, which represent formally the derivatives of all orders of arbitrary functions. Perhaps the basis for a new calculus, founded on the generalized Stokes theorem, is to be looked for here; and this may give access to the relations between differential operators and integral operators. Ideas of this character have already proved very useful in special problems, for instance in the calculus of variations under the name of Haar's lemma as well as in certain papers of Friedrichs. Similarly, the well-known theorem, which asserts that the mean of a harmonic function on a circle is equal to its value at the center, expresses the fact that a certain operator, defined by a mass distribution in the plane, is, in a certain sense, a linear combination of the values of the Laplacian in the closed domain bounded by the circle. Also connected with these questions is the problem, mentioned above, of the representation of differential forms as sums of chains, which arises from the theory of de Rham. It is possible that we have in these researches the dim outlines of an operational calculus, destined to become in one or two centuries as powerful an instrument as the differential calculus has been for our predecessors and for ourselves.

All of this has to do only with the local or semi-local study of partial differential equations; indeed, apart from the simple cases which can be treated by means of the theory of Hilbert spaces or by the direct methods of the calculus of variations, the study of partial differential equations in the large, for instance on a compact analytic manifold, appears too difficult to justify any hope that it can be attacked for a long time to come. On the other hand, the study in the large of ordinary differential equations raises a large number of interesting problems; they are difficult, but within our reach. It will suffice to mention as an example the recent beautiful proof, by E. Hopf, of the ergodic character of the geodesics on every compact Riemannian manifold of everywhere negative curvature. Related to this subject is also the study of van der Pol's equation and of relaxation oscillations, one of the few interesting problems which contemporary physics has suggested to mathematics; for the study of nature, which was formerly one of the main sources of great mathematical problems, seems in recent years to have borrowed from us more than it has given us.

But, incomplete as the foregoing enumeration can not fail to seem to our colleagues, it has undoubtedly exhausted the attention of more than one reader;

and yet, for lack of space and for lack of necessary competence, we have not spoken of the geometry of numbers, nor of diophantine approximations, of the calculus of variations, of the calculus of probabilities, or of hydrodynamics, neither have we mentioned at all several problems, to-day in the background of interest, which could be reactivated by a new idea and restored to the vital stream of mathematics. As a matter of fact, we neither can nor want to lay out a route for the future development of our science; this would be a futile task, indeed it would be a ridiculous enterprise, for the great mathematicians of the future, like those of the past, will flee from the beaten track. They will solve the great problems which we shall bequeath to them, through unexpected connections, which our imagination will not have succeeded in discovering, and by looking at them in a new light. It was our purpose, in passing some of the principal branches of our mathematics in review, to draw attention to their robust vitality and to their fundamental unity. We believe to have shown, not only that there are large numbers of problems, but also that there are very few really important problems which are not intimately related to others which, at first sight, seem to be far removed from them. When a branch of mathematics ceases to interest any but the specialists, it is very near to its death, or at any rate dangerously close to a paralysis, from which it can be rescued only by being plunged back into the vivifying sources of the science. "Mathematics," said Hilbert at the end of his 1900 lecture (and it would be quite in order to quote the conclusion of this lecture in full), "is an organism for whose vital strength the indissoluble union of the parts is a necessary condition."

Does this mean that mathematics is becoming a science for erudites, and that it will no longer be possible to do creative work in mathematics until one has grown gray in the harness, and exhausted from burning the midnight oil for many years in the company of dusty tomes? This would at the same time be a sign of its decline; for, be it strength or weakness, mathematics is not a science that prospers on details, painstakingly collected in the course of a long career, on patient reading, on observations or on filing cards, amassed one by one so as to form a bundle from which an idea will ultimately come forth. Perhaps it is more true in mathematics than in any other branch of knowledge that the idea comes forth in full armor from the brain of the creator. Moreover, mathematical talent usually shows itself at an early age; and the workers of the second rank play a smaller role in it than elsewhere, the role of a sounding board for sounds in whose production they had no part. There are examples to show that in mathematics an old person can do useful work, even inspired work; but they are rare, and each case fills us with wonder and admiration. Therefore, if mathematics is to continue to exist in the way in which it has manifested itself to its votaries until now, the technical complications with which more than one of its subjects is now studded, must be superficial or of only temporary character; in the future, as in the past, the great ideas must be simplifying ideas, the creator must always be one who clarifies, for himself and for others, the most complicated tissues of formulas and concepts. Hilbert indeed asked himself: "Is it not

going to become impossible for the individual worker to embrace all the branches of our science?" and he justified his negative answer, not only by his example, but by remarking that every important advance in mathematics is related to the simplification of methods, to the disappearance of old procedures which have lost their usefulness, and to the unification of branches which were until then foreign to each other. It is quite likely that the contemporaries of Apollonius for example, or those of Lagrange, were familiar with the same feeling of growing complexity which tends to overwhelm us to-day. It is undoubtedly true that the modern mathematician does not know certain details of the theory of conic sections as well as Apollonius did, or as a candidate for a French competitive examination, but this does not lead any one to think that the theory of conic sections should form an autonomous science. Perhaps the same fate is in store for some of the theories of which we are proudest. The unity of mathematics would not be threatened by such an occurrence.

The danger lies elsewhere. Although it is more contingent in character, it does not strike us as less serious; and it seems to us that we cannot bring our reflections on the future of mathematics to a conclusion without saying something of it. We have already said that our civilization itself seems to be under attack from all sides; but this remark was couched in too general terms. "Ne sutor ultra crepidam": it is as mathematicians that we must look at the contemporary world. Our tradition is healthy; are we assured of transmitting it undamaged? In some European countries, particularly in Germany until the start of the Hitler regime, there existed, still a short time ago, university instruction, based on a solid secondary education, which made sure that the mathematical apprentice acquired specific subject-matter knowledge and also the general culture without which nothing of importance can be accomplished. What do we see to-day? In France, none of the essential parts of modern mathematics is taught in our universities, except by a lucky chance. One looks in vain for a university course which puts the advanced student in contact with any one of the great problems which we have listed.* Even the elements of the science are too frequently taught in such a way that the student has to learn everything over again if he wants to push on; the extreme rigidity of a mandarin-caste founded on obsolete academic institutions is the cause of the fact that every attempt at modernization is doomed to failure, unless it remains restricted to verbal changes. Italy, which had formerly a flourishing mathematical school, seems to have fallen into a state of sclerosis analogous to that with which France is threatened, but which has had there still more immediate and destructive effects. We do not know what principles guide, at present, secondary and higher education in the U.S.S.R. There are in that country a number of first-rate mathematicians, but they seem to be absolutely prohibited from crossing the frontiers; and, if such practice should persist, it can hardly in the long run have

* (Author's footnote) This was written in 1946; the same would not be unqualifiedly true to-day.

any other result than the slow choking off of all scientific life. The most remote, as well as the most recent history of our science, shows sufficiently to what extent the contacts between one country and another, prolonged sojourns of students and of teachers at foreign universities, not official sessions at which one drinks toasts while waiting for the next airplane, are an indispensable condition for all progress. We believe that more favorable conditions are found in England and in some of those nations of western Europe which are small only in military statistics. As to Germany, only the future can show whether she will find within herself the necessary elements for linking up with the brilliant tradition interrupted by fifteen years of organized stupidity. Beyond the Atlantic finally, we find a large country, which counts its universities by the hundreds, its students by the hundred thousands, and where, in the words of H. Morrison, the great American specialist in educational problems, "one wanted the education of the masses, one has mass production in education." Thorstein Veblen once sketched in a small book, too little read, the plan of higher education in the United States, and he has done it in a masterly manner; let us merely indicate how the future mathematician is formed in this country which produces more "mathematicians" than perhaps all the rest of the world. In the most favorable cases, one sees a student who, towards the end of his stay at the University, has three or four years at his disposal in which to acquire at the same time the knowledge, the method of work and the elementary intellectual apprenticeship, for which nothing that he has experienced before has in the least prepared him. His only way out under these circumstances is to seek his salvation in the most narrow specialization; in this way, he can often, if he is intelligent and has good guidance, do useful work. Beyond this, he runs the great risk of not being able to survive the stupefying effects of the purely mechanical teaching which he will have to inflict on others, in order to earn his living, after having undergone it himself for too long a time. Whether, in other fields, mass production, thus understood, may produce good results, we are not qualified to determine; we hope to have made it clear that this can not be the case in mathematics. If, unfortunately, the plausible doctrine of making education available to all has had such consequences in a country which lacks, it is true, a strong intellectual tradition, do we not have reason to fear the spread of the contagion to a Europe enfeebled by a catastrophe without precedent?

But if, as Panurge, we ask the oracle questions which are too indiscreet, the oracle will answer us as it did Panurge: Trinck! This advice the mathematician follows gladly, pleased as he is to believe that he will be able to slake his thirst at the very sources of knowledge, convinced as he is that they will always continue to pour forth, pure and abundant, while others have to have recourse to the muddy streams of a sordid reality. If he be reproached with the haughtiness of his attitude, if he be summoned to do his part, if he be asked why he persists on the high glaciers whither no one but his own kind can follow him, he will answer, with Jacobi: For the honor of the human spirit.

THE PROBLEM OF EIGHT POINTS

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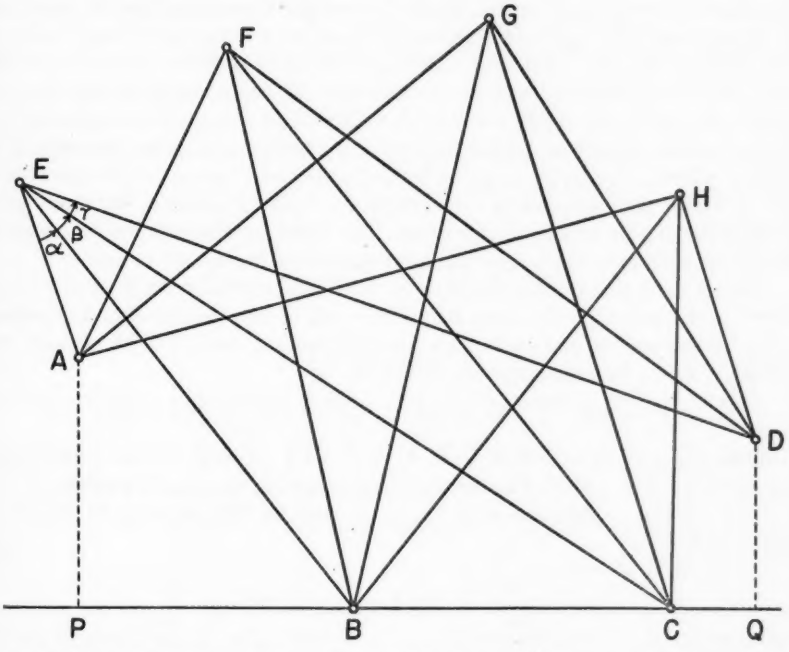
1. Introduction. In a *Treatise on Surveying* by William M. Gillespie* one finds the following problem with figure and comment:



"*Problem of the Eight Points.* Four points, *A*, *B*, *C*, *D*, are inaccessible, but visible from four other points *E*, *F*, *G*, *H;* it is required to find the relative distances of these eight points; the only data being the observation, from each of the points of the second system, of the angles under which are seen the points of the first system.

This problem can be solved, but the great length and complication of the investigation and resulting formulas render it more a matter of curiosity than of utility. It may be found in Puissant's *Topographie*, page 55; Lefevre's *Trigonometrie*, page 90, and Lefevre's *Arpentage*, No. 387."

Puissant gives† a method for handling the problem which is entirely formal and leaves unanswered all the interesting questions with regard to the existence

* Revised by Cady Staley, 1887, p. 256.

† Traite de Topographie, d'Arpentage, et de Nivellement by Puissant, Paris, 1820, p. 55. The problem was brought to my attention by Mr. F. M. Garnett of Savannah, Georgia, who also furnished me with a copy of Puissant's solution.

and number of solutions. The purpose of this paper is to indicate a method of solution (essentially the same as that given by Puissant), to call attention to two Cremona transformations related to the problem, and to give numerical examples of various special cases in which Puissant's treatment breaks down.

2. Puissant's solution. To make the statement of the problem more definite we assume that the eight points are all distinct and lie in a horizontal plane, and we define α as the angle AEB through which an observer at E and facing A must turn to face B, $-180° < \alpha \leqq 180°$, α positive or negative according as the observer turns counter-clockwise or clockwise. At the point E we measure the three such angles, $\alpha = AEB$, $\beta = BEC$, $\gamma = CED$, and at each of the other points of observation the three similar angles; these twelve angles are the only data of the problem. The figure as given by Gillespie and Puissant shows the simple case in which the eight points are vertices of a convex octagon, but there is no assumption of this sort in the problem. The eight points may be anywhere in the plane, with only the restriction that no two of them may coincide.

We set up a rectangular Cartesian coordinate system with B as the origin and C as the point (1, 0), using BC as the unit of length. Let P and Q respectively be the feet of perpendiculars dropped from A and D to the x-axis BC; and let h, k, r, s be four numbers such that

$$|h| = PB, \qquad |k| = PA, \qquad |r| = CQ, \qquad |s| = QD,$$

with the signs so chosen that $(-h, k)$ and $(r+1, s)$ shall be the coordinates, respectively, of A and D. For the figure shown, h, k, r, s are all positive.

Now let the coordinates of E be (u, v), and let the tangents of the given angles at E be

$$\tan\alpha = \frac{a_1}{a_2}, \qquad \tan\beta = \frac{b_1}{b_2}, \qquad \tan\gamma = \frac{c_1}{c_2},$$

with the understanding that a_2, b_2 or c_2 will be zero when the corresponding angle is $\pm 90°$. It can then be shown readily that u, v must satisfy the equations

$$\begin{aligned} &a_1(u^2+v^2) + (a_1h - a_2k)u - (a_1k + a_2h)v = 0 \\ (1)\quad &b_1(u^2+v^2) - b_1u - b_2v = 0 \\ &c_1(u^2+v^2) + (c_2s - c_1r - 2c_1)u - (c_1s + c_2r)v + (c_1r - c_2s + c_1) = 0. \end{aligned}$$

The elimination of u and v from these three equations gives the equation

$$\begin{aligned} &b_1(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1 - a_1b_1c_1)(hr - ks) \\ (2)\qquad &+ b_1(a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2 - a_2b_2c_2)(hs + kr) \\ &- a_1c_1(b_1^2 + b_2^2)(h + r + 1) + a_2c_1(b_1^2 + b_2^2)k + a_1c_2(b_1^2 + b_2^2)s = 0. \end{aligned}$$

The angles α, β, γ being given, this is the condition on h, k, r, s that the equations (1) may have a simultaneous solution for u, v. Geometrically, it is a condition on the situation of the four points A, B, C, D such that they may, at some

point of the plane, subtend the given angles α, β, γ.

Using the given angles at each of the four observation points we have four equations like (2), and we want to solve this set of equations for h, k, r, s. If we set

$$hr - ks = z, \qquad hs + kr = w, \qquad h + r + 1 = t,$$

this set of equations takes the form

$$A_{i1}z + A_{i2}w + A_{i3}t + A_{i4}k + A_{i5}s = 0, \qquad i = 1, 2, 3, 4, \tag{3}$$

linear and homogeneous in z, w, t, k, s.

Puissant then proceeds as follows:

Equations (3) are solved for z, w, t, s as multiples of k, giving

$$z = Zk, \qquad w = Wk, \qquad t = Tk, \qquad s = Sk.$$

From $w = Wk$ and $s = Sk$ we have

$$Shk + kr = Wk, \quad \text{or} \quad Sh + r = W,$$

and from $t = Tk$ we have

$$h + r = Tk - 1.$$

Solving these last two equations for h and r, we substitute the results in

$$z = hr - Sk^2 = Zk$$

and obtain the quadratic equation in k,

$$S\{T^2 + (S-1)^2\}k^2 - \{2ST + TW + STW - Z(S-1)^2\}k + (W+1)(W+S) = 0. \tag{4}$$

Substituting the roots, k and k', of equation (4) in the preceding equations, we obtain two solutions, h, k, r, s and h', k', r', s', of equations (3). We then substitute these sets of values in equations (1), using the tangents of the angles at each of the four observation points, and obtain the coordinates (u, v) of each of these points.

Puissant says that the problem will thus have two solutions; but it is obvious that his procedure may break down at several points. The set of equations (3) may not determine z, w, t, s as multiples of k; equation (4) may be linear rather than quadratic (if $S=0$); equations (1) may not determine u and v uniquely; *etc.* But by far the biggest loop-hole in Puissant's treatment of the problem is due to the fact that the method depends only on the tangents of the twelve given angles.* Hence very often after formal solutions have been found, they will have to be discarded because one or more of the angles of the solution will differ from the given angles by 180°. By the modified problem we shall

* Puissant's equations are expressed in terms of the sines and cosines of these angles, but in such a way that the equations are not altered when any angle is changed by 180°. Hence they depend essentially on the tangents of the angles.

mean the problem in which not the twelve angles, but only their tangents are given. It is this modified problem that Puissant really solves.* Any solution of the problem is of course a solution of the modified problem, but not conversely.

In what follows it is assumed that the twelve given angles (or their tangents) are chosen arbitrarily within the prescribed limits. The problem as worded by Puissant and Gillespie says that these angles are "observed," which would seem to imply the existence of at least one solution; but for angles chosen arbitrarily it will often happen that there are no solutions even for the modified problem (see examples 4, 6, 9, 15, 16 below).

3. Related Cremona transformations. In attempting to make simple numerical examples one discovers two interesting Cremona transformations connected with the problem. Suppose one starts with a set of object points A, B, C, D with rational coordinates, that is, with a rational set of values for h, k, r, s. Rational values are obtained for z, w, t, and then (if $k \neq 0$) for Z, W, T, S, and hence one has the quadratic equation (4) with rational coefficients. One root of this equation is the original k, and hence the other root k' must be rational; and rational values for h', r', s' are then obtained. This means that the set h', k', r', s' is related to the set h, k, r, s by a rational transformation. This transformation is found to be

$$
\begin{aligned}
h' &= \frac{(hs + kr + k)\{r(h + r + 1) - s(k - s)\}}{s\{(h + r + 1)^2 + (k - s)^2\}} \\
k' &= \frac{(hs + kr + k)(hs + kr + s)}{s\{(h + r + 1)^2 + (k - s)^2\}} \\
r' &= \frac{(hs + kr + s)\{h(h + r + 1) + k(k - s)\}}{k\{(h + r + 1)^2 + (k - s)^2\}} \\
s' &= \frac{(hs + kr + s)(hs + kr + k)}{k\{(h + r + 1)^2 + (k - s)^2\}}.
\end{aligned} \tag{5}
$$

This may be regarded as a transformation between points (h, k, r, s) and (h', k', r', s') of a four-dimensional space. It is an involutorial Cremona transformation. Each point in each of the six planes

$$k = s = 0, \quad h = k = 0, \quad r = s = 0, \quad h + 1 = k = 0, \quad r + 1 = s = 0,$$
$$h + r + 1 = k - s = 0,$$

is a fundamental point, and each point of the cubic variety

$$k(r^2 + s^2 + r) - s(h^2 + k^2 + h) = 0$$

is a fixed point.

* Puissant, in his only numerical example, gives "two solutions"; but in one of them the angle γ at E is $-139°$ while the required angle is 41°, so that it is not a solution.

We now have two sets of related object points A, B, C, D and A', B', C', D'. If we next choose any observation point E with rational coefficients, we find rational tangents for the angles α, β, γ at E; and using these tangents with h', k', r', s' in equations (1) we find u', v' as rational coordinates of a related observation point E'. The geometric relationship is that the tangents of the angles α, β, γ subtended by A, B, C, D at E will be the same as the tangents of the angles α', β', γ' subtended by A', B', C', D' at E'. The equations expressing the coordinates (u', v') of E' rationally in terms of the coordinates (u, v) of E are somewhat long and complicated, and it seems hardly worth while to give them here. They represent, however, a Cremona transformation of a well known type,* between points E of a plane π, and points E' of a plane π'. We shall refer to this as transformation (6). The transformation is of the fifth degree with six fundamental points in each plane, A, B, C, D and the two imaginary "circular points" I, J in the plane π, and A', B', C', D' and I', J' in the plane π'. The fundamental curves in each plane are the six conics determined by the fundamental points taken five at a time. This means that the four real fundamental curves in each plane are the four circles determined by A, B, C, D (or A', B', C', D') taken three at a time. Thus if E is taken on the circle ABC, E' will coincide with D', *etc.* (See example 16 below). The properties of the Cremona transformations (5) and (6) correspond to interesting features of the problem of eight points, and enable one to make up rational numerical examples. One chooses A, B, C, D and finds A', B', C', D' by (5); then chooses the observation points E, F, G, H, and for each one (and quite independently of the others) finds the corresponding point E', F', G', or H' by transformation (6); and then finally finds the tangents of the angles α, β, γ at each observation point.

4. Examples of special cases. In the table of numerical examples, page 313, the given angles are indicated by a pair of homogeneous numbers in parentheses preceded by a Roman numeral. Thus II$(-5:3)$ indicates the angle in the second quadrant having the tangent $-5/3$. The quadrantal angles $-90°$, $0°$, $90°$, $180°$ will be designated respectively by III$(1:0)$, IV$(0:1)$, I$(1:0)$, II$(0:1)$. With the restriction $-180° < \alpha \neq 180°$ this notation specifies the angle uniquely. One solves the modified problem ignoring the quadrant specification, obtaining solutions in the form of coordinates for the eight points. In each solution of the modified problem one then checks the angles and discards those solutions in which one or more angles differ from the required angles by $180°$. It is quite obvious that if the twelve given angles are chosen arbitrarily, the chance of obtaining any solution for the problem is small.

Examples 1, 2. These represent the normal case in the sense that Puissant's procedure goes through without trouble, giving in each example two solutions for the modified problem. In the first example there are no solutions for the problem proper because the angle α at G turns out to be $-45°$ instead of $135°$. If we change the problem by specifying the angle IV$(-1:1)$ for α at G, there

* See Cremona Transformations by Hilda P. Hudson, 1927, p. 120.

will be two solutions. In example 2 there is one solution for the problem, but the other fails because α at H is $-135°$ instead of $45°$ as required. No change in the quadrant specification could give us two solutions for this problem, because the angle α at H differs in the two solutions of the modified problem, and if one is right the other must be wrong.

The next few examples indicate ways in which Puissant's handling of the set of equations (3) may break down. In the 4×5 matrix of coefficients of these equations we designate by D_z, D_w, D_t, D_k, D_s respectively the five determinants of fourth order obtained by omitting each column in order. If not all of these determinants vanish, there is homogeneously unique solution for z, w, t, k, s; but we cannot solve for z, w, t, s as multiples of k if $D_k=0$.

Example 3. Here $D_k=0$, $D_s\neq0$. We can solve for z, w, t, k in terms of s, and this leads to a linear equation in s instead of the quadratic equation (4) in k. We find $h=-2$, $k=0$, $r=-2$, $s=1$, and in a strict sense there is only one solution of the modified problem and the problem, namely,

$$A(2, 0),\ B(0, 0),\ C(1, 0),\ D(-1, 1),\ E(0, 1),\ F(1, 2),\ G(-1, -1),\ H(2, -1).$$

But if we make use of transformation (5), keeping h, r, s fixed as above and letting k approach zero, we find that $h'\to-7/5$, $k'\to1/5$, and r' and s' become very large with the ratio $s'/(r'+1)$ approaching $-1/3$. If we then think of D as so far away that the lines of sight from E', F', G', H' to D' all have measurably the same slope, $-1/3$, we have what might be regarded as a second solution

$$A'\left(\frac{7}{5}, \frac{1}{5}\right),\quad B'(0, 0),\quad C'(1, 0),\quad D'\left(\text{slope} -\frac{1}{3}\right),$$

$$E'\left(\frac{2}{5}, \frac{6}{5}\right),\quad F'\left(\frac{3}{2}, \frac{1}{2}\right),\quad G'\left(\frac{7}{5}, -\frac{14}{5}\right),\quad H'\left(\frac{20}{13}, \frac{-4}{13}\right).$$

Examples 4, 5. In example 4, $D_k=D_s=0$, $D_w\neq0$. This gives $k=s=0$, and $w=hs+kr\neq0$, which is impossible, and there is no solution even for the modified problem. In example 5, $D_s=D_k=D_w=0$, $D_z\neq0$. Hence we have $s=k=w=0$, $t=3z$, and it follows that h may be arbitrary and $r=(h+1)/(3h-1)$. We have infinitely many solutions of the modified problem; and we have also infinitely many solutions of the problem proper given by values of h greater than $1/3$. A particularly interesting solution is obtained for $h=1$.

Examples 6, 7. In each of these examples the coefficients of w in equations (3) all vanish, so that we have a set of four homogeneous equations in the four quantities z, t, k, s. In example 6 the determinant does not vanish and the only solution of the set would be $z=t=k=s=0$; but this would make A and D coincide with either B or C and so furnishes no solution of the modified problem. In example 7 the determinant vanishes, and we have

$$z=-\frac{1}{5}k,\qquad t=\frac{8}{5}k,\qquad s=-\frac{1}{5}k,$$

Example		E	F	G	H
1	α	II (−1:2)	IV (−2:1)	II (−1:1)	I (2:3)
	β	III (1:0)	I (1:3)	I (1:1)	IV (−3:11)
	γ	III (2:1)	I (1:2)	IV (−1:2)	I (1:2)
2	α	I (4:3)	I (21:25)	I (1:4)	I (1:1)
	β	I (1:4)	I (1:5)	I (1:13)	I (1:9)
	γ	I (7:6)	I (1:1)	III (3:1)	I (11:24)
3	α	IV (−2:1)	IV (−4:3)	I (1:2)	I (2:1)
	β	I (1:1)	I (1:2)	IV (−1:3)	IV (−1:3)
	γ	III (1:1)	IV (−2:1)	I (2:1)	I (1:5)
4	α	IV (−1:3)	I (1:1)	I (1:2)	IV (−1:1)
	β	I (1:2)	IV (−1:3)	I (1:1)	I (2:1)
	γ	IV (−4:3)	I (1:3)	IV (−14:13)	IV (−12:1)
5	α	I (1:1)	I (1:1)	I (1:3)	I (1:7)
	β	I (1:3)	I (1:1)	I (1:1)	I (1:3)
	γ	I (1:7)	I (1:3)	I (1:1)	I (1:1)
6	α	IV (−1:4)	I (1:2)	I (1:1)	I (8:7)
	β	I (2:1)	I (1:1)	I (5:9)	I (20:39)
	γ	I (6:7)	I (1:3)	I (2:7)	I (1:4)
7	α	I (1:3)	I (7:11)	I (17:31)	I (11:23)
	β	I (1:1)	IV (−1:4)	I (1:7)	I (5:12)
	γ	I (1:2)	III (3:1)	I (4:3)	I (17:19)
8	α	II (−8:1)	I (2:1)	I (8:9)	I (8:19)
	β	I (6:17)	I (1:2)	I (2:5)	I (14:73)
	γ	I (2:1)	I (1:0)	II (−2:1)	II (−2:3)
9	α	I (1:7)	I (1:7)	I (1:3)	I (1:1)
	β	I (3:4)	I (3:4)	I (3:2)	II (−3:2)
	γ	I (1:2)	I (1:2)	I (2:3)	I (1:1)
10	α	I (1:7)	I (1:7)	I (1:3)	I (1:6)
	β	I (3:4)	I (3:4)	I (3:2)	I (6:1)
	γ	I (1:2)	I (1:2)	I (2:3)	II (−2:3)
11	α	II (−1:2)	IV (−2:1)	III (9:1)	IV (−1:2)
	β	III (1:0)	I (1:3)	I (1:1)	I (13:51)
	γ	III (2:1)	I (1:2)	I (1:3)	I (1:60)
12	α	I (1:2)	I (3:4)	IV (−1:4)	I (5:12)
	β	I (2:3)	I (4:3)	IV (−2:3)	I (2:3)
	γ	I (11:16)	I (7:6)	IV (−7:22)	I (3:2)
13	α	I (1:18)	I (1:3)	I (1:6)	IV (−1:1)
	β	III (12:5)	I (3:2)	I (6:1)	IV (−3:4)
	α	I (8:27)	I (2:3)	II (−2:3)	I (13:34)
14	α	I (1:7)	I (1:3)	I (1:6)	IV (−1:1)
	β	I (3:4)	I (3:2)	I (6:1)	IV (−3:4)
	γ	I (1:2)	I (2:3)	II (−2:3)	I (13:34)
15	α	I (173:29)	I (33:1)	I (21:2)	I (33:17)
	β	I (3:1)	I (2:1)	I (1:1)	I (1:0)
	γ	I (1:1)	I (1:1)	I (2:1)	I (1:1)
16	α	IV (−3:4)	I (5:2)	I (1:0)	I (1:1)
	β	I (1:3)	IV (−3:7)	IV (−1:1)	IV (−1:3)
	γ	IV (−1:7)	III (1:1)	II (−1:3)	III (1:3)

which leads to the equation

$$5r^2 - (8k - 5)r - (k^2 + k) = 0.$$

Choosing an arbitrary value for k we get two values for r and h, and hence two solutions for the modified problem. We will therefore have an infinite number of solutions of the modified problem, and in this particular example also an infinite number of solutions of the problem proper. Two fairly simple solutions are given by taking $k=1$ with $r=1$ and $k=5/7$ with $r=4/7$. The vanishing of all the coefficients of w in this example corresponds to the geometric fact that $\alpha+\beta+\gamma=\pm 90°$ at each of the observation points.

Example 8. All the coefficients of z in the equations (3) vanish, giving four equations in w, t, k, s, and the determinant is of rank 2. We may choose both k and s arbitrarily, and then

$$h = -2k + 4, \qquad r = 2s - 5,$$

which means that A and D lie anywhere on the line $x-2y+4=0$. It is then found that the four observation points also lie on this line at fixed positions for all choices of k and s. The vanishing of the coefficients of z means that at each observation point $\alpha+\beta+\gamma=0°$ or 180°, and hence each observation point must lie on the line AD. There are infinitely many solutions for the modified problem; and also, if we take $k<3/2$ and $s>7/2$, infinitely many solutions for the problem.

Examples 9, 10. The angles at E are exactly the same as the angles at F, so that the first two of the four equations (3) are the same, and hence $D_z=D_w=D_t=D_k=D_s=0$. It is easy to show, however, that the angles at two distinct points E and F could be the same only if A, B, C, D, E, F all lie on one circle. Using this fact and the tangents of the angles at E and F (and abandoning Puissant's procedure), we deduce without difficulty that

$$h = \frac{1}{6}, \qquad k = \frac{1}{6}, \qquad r = \frac{1}{3}, \qquad s = \frac{2}{3},$$

and hence

$$z = -\frac{1}{18}, \qquad w = \frac{1}{6}, \qquad t = \frac{3}{2}.$$

These values may or may not satisfy the two remaining equations of set (3). In example 9 the last equation is not satisfied, and hence there are no solutions even for the modified problem. In example 10 the last two equations are satisfied There is only one possible position for the points A,B,C,D (on a circle), but there are infinitely many solutions of the modified problem because E and F may be taken anywhere on the circle. The three equations of set (1) become identical for the points E and F. There are infinitely many solutions of the problem proper with E and F restricted to certain arcs of the circle.

Example 11. For the set of equations (3) we have $D_z = D_w = D_t = D_k = D_s = 0$ without any situation quite so obvious as in examples 7, 8, 9, 10. The matrix is of rank 3, and we can solve for z, w, t in terms of k and s,

$$z = \frac{-8k + 13s}{9}, \qquad w = \frac{2k - 19s}{9}, \qquad t = \frac{-5k + 7s}{9}.$$

We can then give arbitrary values to S in the relation $s = Sk$, and proceed with Puissant's method, obtaining (usually) two solutions of the modified problem for each value assigned to S. There are infinitely many solutions of both the modified problem and the problem. For $S = 2$ we get two fairly simple solutions of the modified problem, both of which are solutions of the problem.

In the remaining examples the determinant D_k for the set of equations (3) does not vanish, and there is a unique solution for z, w, t, s as multiples of k; also we can set up the quadratic equation (4). For this equation the discriminant may be positive, negative, or zero, and Puissant ignores the last two possibilities. The discriminant vanishes under either of the two following geometric conditions: when AD is parallel to BC so that $s = k$ ($S = 1$), and when A, B, C, D all lie on one circle for which the condition is

$$k(r^2 + s^2 + r) - s(h^2 + k^2 + h) = 0. \tag{7}$$

Example 12. We have

$$z = k, \qquad w = 2k, \qquad t = 6k, \qquad s = k,$$

and equation (4) becomes

$$36k^2 - 36k + 9 = 0,$$

giving $\frac{1}{2}$ as the only value for k and for s. But there are two values each for h and r, and we have the two solutions for equations (3),

$$h = \frac{1}{2}, \qquad k = \frac{1}{2}, \qquad r = \frac{3}{2}, \qquad s = \frac{1}{2}$$

$$h' = \frac{3}{2}, \qquad k' = \frac{1}{2}, \qquad r' = \frac{1}{2}, \qquad s' = \frac{1}{2}.$$

This leads to two solutions for both the modified problem and the problem.

Examples 13, 14. In both of these examples we have

$$z = -\frac{1}{3}k, \qquad w = k, \qquad t = 9k, \qquad s = 4k,$$

and equation (4) is

$$360k^2 - 120k + 10 = 0,$$

giving $k = 1/6$ and the one set of values

$$h = \frac{1}{6}, \qquad k = \frac{1}{6}, \qquad r = \frac{1}{3}, \qquad s = \frac{2}{3}.$$

These values satisfy the relation (7), showing that A, B, C, D lie on a circle. In example 13 the observation points are uniquely determined (none of them lying on the circle $ABCD$), and we have one solution for the problem. But in example 14, when we try to find the coördinates of E, the three equations of set (1) are all the same as the equation of the circle $ABCD$, and E is any point on this circle giving infinitely many solutions of the modified problem. There are also infinitely many solutions of the problem proper, with E on the major arc AD of the circle.

Example 15. Here we find

$$z = 24k, \qquad w = 8k, \qquad t = 3k, \qquad s = 2k,$$

and equation (4) is

$$20k^2 - 60k + 90 = 0.$$

The values of k are imaginary, and therefore there are no solutions even for the modified problem.

Example 16. Puissant's method goes through, giving what would be two solutions of the modified problem except for the fact that in each solution an observation point coincides with an object point, F with B in one solution and G' with D' in the other. In a strict sense we have no solutions. But it is possible to take a certain line through F (the line with slope 3/7) as the line of sight from F to B, and similarly a line through G' (with slope 1) as the line $G'D'$, giving two pseudo-solutions of the modified problem. In terms of the Cremona transformation (6), G is on the fundamental circle ABC in the plane π, and F' is on the fundamental circle $A'C'D'$ in the plane π', so that G' and F coincide respectively with the fundamental points D' and B.

The interested reader will find it not too difficult to make up examples of other cases for which the Puissant procedure will break down, and for which there may be no solutions, one, or infinitely many.

FACTORIZATION OF POLYNOMIALS

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1. Introduction. Let D be an integral domain* and let $D[x]$ be the integral domain of polynomials in x with coefficients in D. Then

THEOREM. *If D is a unique factorization domain (abbreviated UFD), $D[x]$ is also.*

It follows from this theorem, by mathematical induction on n, that when D is a UFD then the domain $D[x_1, x_2, \cdots, x_n]$ of polynomials in $x_1, x_2, \cdots, x_n$ with coefficients in D is also a UFD.

In the usual proof† of the theorem, the desired result is first established when D is a field. The proof of this depends upon:

(a) the division algorithm, which is used to prove
(b) if $f(x)$ and $g(x)$ are non-zero polynomials in $D[x]$, there exists a greatest common divisor (abbreviated GCD) $d(x)$ expressible in the form $a(x)\,f(x) + b(x)\,g(x)$, where $a(x)$ and $b(x)$ are elements of $D[x]$, which is used to prove
(c) if a prime polynomial divides the product of two polynomials it divides one of the factors. Using this special case of the theorem and
(d) Gauss' lemma and
(e) the existence of a quotient field for any integral domain, the general case of the theorem is established.

The usual proof of the fact that the integers form a UFD also makes use of results similar to (a), (b) and (c). However, E. Zermelo‡ has given a very simple proof which avoids these considerations. It is the purpose of this note to present a similar proof of the theorem above which does not require the use of (a)–(e).

2. Existence of factorization. Let D be a UFD. We show that every non-zero element $f(x)$ of $D[x]$ which is not a unit is expressible in the form $p_1(x)p_2(x)\cdots p_k(x)$, $k \geqq 1$, where the $p_i(x)$ are primes in $D[x]$.

We shall make use of the following easily demonstrable facts:

(1) The units of $D[x]$ are the units of D.
(2) If a constant divides a polynomial it divides every coefficient of the polynomial.
(3) A constant is prime in $D[x]$ if and only if it is prime in D.
(4) If $a_1, a_2, \cdots, a_k$ $(k \geqq 2)$ are elements of D not all zero, there exists a GCD a in D, and if $a_i = ab_i$ $(i = 1, 2, \cdots, k)$ then the b_i have no common factors in D except units.

Now let $f(x)$ be of degree n. If $n = 0$, $f(x)$ is in D. Since it is not a unit of $D[x]$, it is not a unit of D. Hence it is a product of primes in D which are also primes in $D[x]$.

* For definitions of terms see Birkhoff-MacLane, A Survey of Modern Algebra, Macmillan, 1941.

† Birkhoff-MacLane, pp. 91–100.

‡ Nachr. Ges. Wiss. Göttingen I, N.F. 1, 1934, pp. 43–46. The complete proof will also be found in a few lines in Zentralblatt 10, 1935, p. 293.

Proceeding by induction on n, let $n>0$ and suppose the desired result established for polynomials of degree $0, 1, \cdots, n-1$.

Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \qquad n > 0,\ a_0 \neq 0$$
$$a = \text{GCD of } a_0, a_1, \cdots, a_n$$
$$a_i = ab_i \qquad (i = 0, 1, \cdots, n)$$
$$g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n.$$

Then

$$f(x) = ag(x).$$

Since $a \neq 0$ is in D, it is a unit or is factorable into primes.

If $g(x)$ is a prime, the desired result is established (using, if necessary, the fact that the product of a unit and a prime is a prime). Otherwise, there exists a factorization $g(x)=s(x)t(x)$, where $s(x)$ and $t(x)$ are elements of $D[x]$ neither of which is a unit. Neither $s(x)$ nor $t(x)$ can be a constant. For, if either were a constant, it would have to divide each of the coefficients of $g(x)$, and this is impossible since the b_i have no common divisors except units. Thus, each of $s(x)$ and $t(x)$ is of degree less than n. Therefore, by the hypothesis of the induction, each is factorable into primes, and the desired result is established.

3. Uniqueness of factorization. We show that if

$$f(x) = p_1(x)p_2(x) \cdots p_k(x) = q_1(x)q_2(x) \cdots q_l(x),$$

where the $p_i(x)$ and $q_j(x)$ are primes, then $k=l$ and the $p_i(x)$ can be paired with the $q_j(x)$ in such a way that the $p_i(x)$ and $q_j(x)$ in each pair are associates.

If $n=0$ then $f(x)$ and the prime factors are in D. Since D is a UFD, the desired results is immediate. Proceeding by induction on the degree of $f(x)$, let $n>0$, and suppose the desired result established for polynomials of degree $0, 1, \cdots, n-1$.

If $k=l=1$, there is nothing more to be proved.

If $k=1$ and $l>1$, then $q_1(x)[q_2(x) \cdots q_l(x)]$ is a prime. Hence $q_2(x) \cdots q_l(x)$ is a unit, so that $q_2(x)$ is a unit. This is impossible.

Similarly, $k>1$ and $l=1$ is impossible.

Let

$$k > 1, \qquad l > 1$$
$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \qquad n > 0,\ a_0 \neq 0$$
$$p_i(x) = a_{i0}x^{n_i} + \cdots + a_{in_i},\ a_{i0} \neq 0 \qquad (i = 1, 2, \cdots, k),$$
$$q_j(x) = b_{j0}x^{m_j} + \cdots + b_{jm_j},\ b_{j0} \neq 0 \qquad (j = 1, 2, \cdots, l).$$

Let n_1 be one of the smallest of the n_i, and let m_1 be one of the smallest of the m_j. Either $n_1 \leqq m_1$ or $m_1 \leqq n_1$. Suppose, to be specific, $n_1 \leqq m_1$. Then $n_1 \leqq m_j$ for

$j=1, 2, \cdots, l$.

We show that $p_1(x)$ is an associate of some $q_j(x)$.

If $n_1=0$,* then $p_1(x)=a_{10}$ is a prime in D which is a factor of

$$a_0 = a_{10}a_{20}\cdots a_{k0} = b_{10}b_{20}\cdots b_{l0}.$$

Since D is a UFD, a_{10} is an associate of one of the primes in the factorizations of $b_{10}, b_{20}, \cdots, b_{l0}$. Therefore one of $b_{10}, b_{20}, \cdots, b_{l0}$ is divisible by a_{10}. Let

$$\begin{aligned} b_{u0} &= a_{10}c \\ P(x) &= q_1(x)\cdots q_{u-1}(x)p_1(x)q_{u+1}(x)\cdots q_l(x) \\ &= (b_{10}\cdots b_{u-1,0}a_{10}b_{u+1,0}\cdots b_{l0})x^m + \text{lower powers of } x, \text{ if any,} \end{aligned}$$

where

$$\begin{aligned} m &= m_1 + \cdots + m_{u-1} + n_1 + m_{u+1} + \cdots + m_l \\ &\leqq m_1 + \cdots + m_{u-1} + m_u + m_{u+1} + \cdots + m_l = n. \end{aligned}$$

Let $R(x)=f(x)-cx^{n-m}P(x)$. Then either $R(x)=0$, or the degree of $R(x)$ is less than n.

If $n_1>0$, let u be any integer between 1 and l inclusive and define $P(x)$ as above. Let

$$R(x) = a_{10}f(x) - b_{u0}x^{n-m}P(x).$$

Again either $R(x)=0$ or the degree of $R(x)$ is less than n. Combining both cases we have

$$R(x) = \alpha f(x) - Q(x)P(x)$$

where $Q(x)$ is in $D[x]$, $\alpha\neq 0$ is in D, and $\alpha=1$ if $p_1(x)$ is a constant.

We consider two cases:

(a) $R(x)=0$. Then

$$\begin{aligned} \alpha f(x) &= \alpha q_1(x)q_2(x)\cdots q_l(x) \\ &= Q(x)q_1(x)\cdots q_{u-1}(x)p_1(x)q_{u+1}(x)\cdots q_l(x), \end{aligned}$$

so that

$$\alpha q_u(x) = q(x)p_1(x).$$

If $n_1=0$ then $\alpha=1$ and $p_1(x)$, $q_u(x)$ are associates.

If $n_1>0$ then every $m_j\geqq n_1>0$. Since $l\geqq 2$, therefore every $m_j<n$. Hence $m_u<n$. Therefore, by the hypothesis of the induction, $\alpha q_u(x)$ is uniquely factorable into primes. By factoring α (unless it is a unit) we see that one of the primes is $q_u(x)$ or an associate and the others are constants. By factoring $Q(x)$ (unless it is a unit) we see that one of the primes is $p_1(x)$ or an associate. Hence $p_1(x)$ is an associate of $q_u(x)$ or of a constant. Since $n_1>0$, $p_1(x)$ is not a constant. Therefore it is an associate of $q_u(x)$.

* This cannot happen if D is a field.

(b) $R(x) \neq 0$. We have

$$R(x) = \alpha q_1(x)q_2(x) \cdots q_l(x) - Q(x)q_1(x) \cdots q_{u-1}(x)p_1(x)q_{u+1}(x) \cdots q_l(x)$$
$$= q_1(x) \cdots q_{u-1}(x)q_{u+1}(x) \cdots q_l(x)[\alpha q_u(x) - Q(x)p_1(x)].$$

Also

$$R(x) = \alpha p_1(x)p_2(x) \cdots p_k(x) - Q(x)q_1(x) \cdots q_{u-1}(x)p_1(x)q_{u+1}(x) \cdots q_l(x)$$
$$= p_1(x)S(x).$$

$R(x)$ is not a unit, since $p_1(x)$ is a factor of $R(x)$. Therefore, by the hypothesis of the induction, $R(x)$ is uniquely factorable into primes. Its prime factors are $q_1(x), \cdots, q_{u-1}(x), q_{u+1}(x), \cdots, q_l(x)$ and the prime factors of $\alpha q_u(x) - Q(x)p_1(x)$. Also, its prime factors are $p_1(x)$ and the prime factors of $S(x)$. Therefore, $p_1(x)$ is an associate of one of $q_1(x), \cdots, q_{u-1}(x), q_{u+1}(x), \cdots, q_l(x)$ or of one of the prime factors of $\alpha q_u(x) - Q(x)p_1(x)$. In the latter case $p_1(x)$ is a factor of $\alpha q_u(x) - Q(x)p_1(x)$, so that

$$\alpha q_u(x) - Q(x)p_1(x) = T(x)p_1(x)$$
$$\alpha q_u(x) = [Q(x) + T(x)]p_1(x).$$

It now follows as in case (a) that $p_1(x)$ is an associate of $q_u(x)$.
Thus, in all cases, $p_1(x)$ is an associate of $q_u(x)$.

If $p_1(x) = \epsilon q_v(x)$ where ϵ is a unit, then $p_1(x)p_2(x) \cdots p_k(x) = q_1(x)q_2(x) \cdots q_l(x)$ yields

$$[\epsilon p_2(x)]p_3(x) \cdots p_k(x) = q_1(x) \cdots q_{v-1}(x)q_{v+1}(x) \cdots q_l(x)$$

where $\epsilon p_2(x)$ is a prime.

If the degree of $\epsilon p_2(x)p_3(x) \cdots p_k(x)$ is less than n, the desired result follows by the hypothesis of the induction. If not, then we may again proceed as above.

Since the process reduces the number of primes, we must, after applying it a sufficient number of times, arrive at a stage where either the hypothesis of the induction applies or else there is only one prime on one side of the equality. In either case the desired result follows.

MATHEMATICAL NOTES

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ON THE LATENT ROOTS OF QUASI-COMMUTATIVE MATRICES

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We consider matrices A, B of order n such that

$$AB = \omega BA,$$

where ω is a primitive qth root of unity, $\omega^q=1$. Thus B is a commutant[1] of A and ωA, so that, if $B\neq 0$, A and ωA must have a common latent root[1]; that is, for at least one pair α, β of latent roots of A we have $\beta=\omega\alpha$. If B is non-singular, since $B^{-1}AB=\omega A$, the latent roots of A are clearly of the form α_1, $\omega\alpha_1$, $\omega^2\alpha_1$, $\cdots$, $\omega^{q-1}\alpha_1$, α_2, $\omega\alpha_2$, $\cdots$. It is simple to prove the following result.

Theorem 1. *If α is a latent root of A, x a corresponding latent vector such that $Bx\neq 0$, then $\omega\alpha$ is also a latent root of A.*

Proof. Put $y=Bx$; then we have

$$Ay = ABx = \omega BAx = \omega\alpha Bx = \omega\alpha y,$$

since $Ax=\alpha x$. Also $y\neq 0$; hence $\omega\alpha$ is a latent root of A.

On the other hand we have the example

$$A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here $AB=-BA$, α and $-\alpha$ are latent roots of A, and yet $Bx=0$ for each latent vector x of A.

Theorem 2. *The latent roots, $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$, of A, B may be taken in such an order that the latent roots of*

(i) $$A + B \text{ are } (\alpha_r^q + \beta_r^q)^{1/q} \qquad (r = 1, \cdots, n),$$

and of

(ii) $$AB \text{ are } \omega^{\frac{1}{2}(q-1)}\alpha_r\beta_r \qquad (r = 1, \cdots, n).$$

[1] Cf. H. W. Turnbull, and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices*, London, 1932, p. 148.

Proof. Let

$$(A + B)^k = \sum_{r=0}^{k} c_r^{(k)} B^r A^{k-r},$$

so that $c_0^{(k)} = c_k^{(k)} = 1$. Since $(A+B)^{k+1} = (A+B)(A+B)^k = (A+B)^k(A+B)$, we have

$$\sum_{r=0}^{k+1} c_r^{(k+1)} B^r A^{k+1-r} = \sum_{r=0}^{k} \omega^r c_r^{(k)} B^r A^{k+1-r} + \sum_{r=0}^{k} c_r^{(k)} B^{r+1} A^{k-r}$$

$$= \sum_{r=0}^{k} c_r^{(k)} B^r A^{k+1-r} + \sum_{r=0}^{k} \omega^{k-r} c_r^{(k)} B^{r+1} A^{k-r};$$

for $A^r B^s = \omega^{rs} B^s A^r$. Thus

$$c_r^{(k+1)} = \omega^r c_r^{(k)} + c_{r-1}^{(k)} = c_r^{(k)} + \omega^{k+1-r} c_{r-1}^{(k)},$$

and we deduce that this generalized binomial coefficient $c_r^{(k)}$ has the value

$$c_r^{(k)} = \frac{\phi_k}{\phi_r \phi_{k-r}}, \quad \text{where} \quad \phi_r = \prod_{s=1}^{r} (\omega^s - 1).$$

In particular, if ω is a primitive qth root of unity, we have $c_r^{(q)} = 0$, for $r = 1, 2, \cdots, q-1$, and

$$(A + B)^q = A^q + B^q.$$

By induction, $(AB)^k = \omega^{k(1-k)/2} A^k B^k$. As $A^q B^q = B^q A^q$, we have by Frobenius' theorem[2] that, if $f(x, y)$ is any rational function, then the latent roots of $f(A^q, B^q)$ are $f(\alpha_r^q, \beta_r^q)$ $(r = 1, \cdots, n)$ for some pairing of the roots α_r, β_r, and this is the same for every function $f(x, y)$. Thus the latent roots of $(A+B)^q$, $(AB)^q$ are

$$\alpha_r^q + \beta_r^q, \qquad \omega^{\frac{1}{2}q(q-1)} \alpha_r^q \beta_r^q \qquad (r = 1, \cdots, n).$$

Hence the latent roots of $A+B$ are $(\alpha_r^q + \beta_r^q)^{1/q} (r = 1, \cdots, n)$, and those of AB are $\omega^{\frac{1}{2}(q-1)} \alpha_r \beta_r$ $(r = 1, \cdots, n)$.

A CRITERION FOR A COMMON ROOT OF k ALGEBRAIC EQUATIONS

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1. Introduction. A necessary and sufficient condition that two algebraic equations $f(x) = 0$ and $g(x) = 0$ have a common root is the vanishing of Sylvester's determinant $R(f, g)$. But there seems to exist no simple criterion that more than two equations have a common root. In this paper such a criterion will be obtained under the assumption that at least one of the equations has only real roots.

The origin of this paper was Problem 2 of the Mathematical Series of the

[2] Frobenius, Berliner Sitzungsberichte, I (1896), pp. 601–614.

Rand Corporation in Santa Monica; the statement of this problem follows:

Given $n+1$ polynomials of degree n, every n of which are linearly independent, and each of which is non-negative for $0 \leqq x \leqq 1$ and has n roots (counting multiplicities) in that interval. Is it true that these polynomials have no root in common if, and only if, all $n+1$ are linearly independent? If not true, what additional conditions are necessary for the $n+1$ polynomials to have a common root?

2. Solution of problem 2. Let $f_1, f_2, \cdots, f_{n+1}$ be $n+1$ arbitrary polynomials of degree n. If these polynomials have a common root, then it is necessary that they are linearly dependent. This follows at once from the fact that they can be considered as linear forms in $1, x, x^2, \cdots, x^n$.

The converse is not correct even if the polynomials are non-negative in the interval $0 \leqq x \leqq 1$ and have there all their roots. This is shown by the example:

$$f_1(x) = (x - 1/2)^2(x - 1/4)^2, \qquad f_2(x) = (x - 1/3)^4,$$
$$f_3(x) = x^2(x - a)^2, \qquad f_4(x) = x^2(x - b)^2, \qquad f_5(x) = x^2(x - c)^2,$$

where $0 \leqq a \leqq 1$, $0 \leqq b \leqq 1$, $0 \leqq c \leqq 1$. They are linearly dependent since

$$\begin{vmatrix} & & \cdot & -\frac{3}{16} & \frac{1}{64} \\ & & \cdot & & \\ & & \cdot & -\frac{4}{27} & \frac{1}{81} \\ & & \cdot & & \\ \cdots & \cdots & \cdot & \cdots & \cdots \\ & & \cdot & 0 & 0 \\ & & \cdot & & \\ & \Delta & \cdot & 0 & 0 \\ & & \cdot & & \\ & & \cdot & 0 & 0 \end{vmatrix} = \frac{\Delta}{16 \cdot 27} \begin{vmatrix} 3 & \frac{1}{4} \\ 4 & \frac{1}{3} \end{vmatrix} = 0.$$

The following condition is sufficient and necessary. Let f be one of the $n+1$ polynomials and F the sum of the n other polynomials. The polynomials have a common root if and only if f and F have a common root; that is, if and only if Sylvester's determinant $R(f, F)$ satisfies $R(f, F) = 0$.

Proof: If the polynomials have a common root, then this root is a common root of f and F. Conversely, assume that f and F have a common root. Then this root is a root of f and therefore real and lies in the interval $0 \leqq x \leqq 1$. On the other hand, F is non-negative in this interval and vanishes there only if all the terms of F have a common root. Hence all the polynomials have a common root.

Sometimes the computation of the resultant $R(f, F)$ can be simplified by using $c_\nu f_\nu(x)$ with positive c_ν instead of $f_\nu(x)$.

3. Generalization. This result can be generalized as follows.

THEOREM. *Let $f_1, f_2, \cdots, f_k$ be polynomials of arbitrary degree with real or complex coefficients. Set*

$$f_\kappa(x) = g_\kappa(x) + ih_\kappa(x) \qquad (\kappa = 1, 2, \cdots, k), \tag{1}$$

where $g_\kappa(x)$ *and* $h_\kappa(x)$ *have real coefficients. Moreover, set*

$$\bar{f}_\kappa(x) = g_\kappa(x) - ih_\kappa(x) \qquad (\kappa = 1, 2, \cdots, k).$$

Assume that one of these polynomials, say f_1, *has only real roots. Set*

$$F(x) = \sum_{\kappa=1}^{k} f_\kappa(x)\bar{f}_\kappa(x) = \sum_{\kappa=1}^{k} g_\kappa^2(x) + h_\kappa^2(x). \tag{2}$$

A necessary and sufficient condition that the polynomials (1) *have a common root is that* f_1 *and* F *have a common root.*

Proof. If the polynomials (1) have a common root ξ, then ξ is a root of $F(x)$ by (2). Conversely, if f_1 and F have a common root ξ, then ξ is real since it is a root of f_1. But F is non-negative for real values of x and vanishes only if all the polynomials $g_\kappa(x)$ and $h_\kappa(x)$ have a common root. Hence

$$g_\kappa(\xi) = h_\kappa(\xi) = 0 \qquad (\kappa = 2, 3, \cdots, k).$$

It follows that ξ is a common root of the polynomials (1).

ON A DETERMINANT OF SYLVESTER*

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1. Introduction. Let $\epsilon = e^{2\pi i/n}$. Then the element in the rth row and cth column of the determinant V of order n which is the subject of this note is defined by $v_{rc} = \epsilon^{(r-1)(c-1)}$. The first known application of this interesting determinant is due to Sylvester [1], who gave a formula for the value of V. In this note we show that V is reducible and obtain the values of the two component determinants; their elements contain only cosines and sines, respectively.

2. Discussion. Consider the transformation

$$x_j = \sum_{k=1}^{n} \epsilon^{(j-1)(k-1)} x_k', \qquad j = 1, \cdots, n, \tag{1}$$

of determinant V. This may be written as

$$x_{2-j} = \sum_{k=1}^{n} \epsilon^{(1-j)(k-1)} x_k', \qquad i = 1, \cdots, n, \tag{2}$$

where it is assumed that n is added to any subscript not in the range $1, \cdots, n$. The coefficients of x_k' in (1) and (2) are the same conjugate complex numbers as the coefficients of x_{2-k}' in (2) and (1). For an odd $n > 1$, we obtain, from (1) and (2),

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$$\frac{1}{2}(x_j + x_{2-j}) = x_1' + \sum_{k=2}^{(n+1)/2} 2\cos(j-1)(k-1)\frac{2\pi}{n}\cdot\frac{1}{2}(x_k' + x_{2-k}'), \qquad j = 1, \cdots, \frac{n+1}{2};$$

(3)

$$\frac{1}{2}(x_j - x_{2-j}) = \sum_{k=2}^{(n+1)/2} 2i\sin(j-1)(k-1)\frac{2\pi}{n}\cdot\frac{1}{2}(x_k' - x_{2-k}'), \qquad j = 2, \cdots, \frac{n+1}{2}.$$

Introducing new variables by means of the non-singular transformation,

$$y_j = \frac{1}{2}(x_j + x_{2-j}), \qquad j = 1, \cdots, \frac{n+1}{2},$$

(4)

$$y_{(n-1)/2+j} = \frac{1}{2}(x_j - x_{2-j}), \qquad j = 2, \cdots, \frac{n+1}{2},$$

of determinant T, we obtain

$$y_j = y_1' + \sum_{k=2}^{(n+1)/2} 2\cos(j-1)(k-1)\frac{2\pi}{n}\cdot y_k', \qquad j = 1, \cdots, \frac{n+1}{2},$$

(5)

$$y_{(n-1)/2+j} = \sum_{k=2}^{(n+1)/2} 2i\sin(j-1)(k-1)\frac{2\pi}{n}\cdot y_{(n-1)/2+k}', \qquad j = 2, \cdots, \frac{n+1}{2}.$$

The determinant of the transformation (5) is given by TVT^{-1}.

The elements of V^2 are defined by $n\delta'_{2-c}$, where $\delta'_{2-c}=1$, if $r\equiv 2-c \pmod n$ and $\delta'_{2-c}=0$, if $r\not\equiv 2-c \pmod n$. The only non-zero elements of TV^2T^{-1} are in the diagonal. The first $(n+1)/2$ of these are n and the last $(n-1)/2$ are $-n$.

Let A and B represent the determinants of (5) considered as two transformations in the order in which they are written. Then

$$A^2 = n^{(n+1)/2}, \qquad B^2 = (-n)^{(n-1)/2}.$$

If in A we add the elements of the remaining columns to the elements of the first column, the first element in the new first column is n and the remaining elements are zero. Hence $A_1^2 = n^{(n-3)/2}$, where A_1 represents the minor of the element in the upper left-hand corner of A.

If we make the necessary adjustments for an even $n>2$, it may be shown that the minor corresponding to A_1 is defined by

$$a_{rc} = 2\cos rc\frac{2\pi}{n}, \qquad r = 1, \cdots, \frac{n}{2},$$

(6)

$$a_{r(n/2)} = (-1)^r, \qquad c = 1, \cdots, \frac{n}{2} - 1.$$

If square roots are extracted, a result due to J. S. Frame [2] may be used to select the proper sign.

3. Results. We have thus established the following theorems.

THEOREM 1. *If* $n=4k+1$, *then we have the following values of determinants*:

$$\left|2\cos rc\frac{2\pi}{n}\right| = (-1)^k n^{(n-3)/4},$$

$$\left|2i\sin rc\frac{2\pi}{n}\right| = n^{(n-1)/4}, \qquad r, c = 1, \cdots, \frac{n-1}{2}.$$

If $n=4k+3$, *then*

$$\left|2\cos rc\frac{2\pi}{n}\right| = (-1)^{k+1} n^{(n-3)/4},$$

$$\left|2i\sin rc\frac{2\pi}{n}\right| = in^{(n-1)/4}, \qquad r, c = 1, \cdots, \frac{n-1}{2}.$$

THEOREM 2. *If* $n=4k$, *the determinant defined* by (6) *equals* $(-1)^k n^{(n-2)/4}$; *also*

$$\left|2i\sin rc\frac{2\pi}{n}\right| = in^{(n-2)/4}, \qquad r, c = 1, \cdots, \frac{n}{2}-1.$$

If $n=4k+2$, *the determinant defined* by (6) *equals* $(-1)^{k+1} n^{(n-2)/4}$; *also*

$$\left|2i\sin rc\frac{2\pi}{n}\right| = n^{(n-2)/4} \qquad r, c = 1, \cdots, \frac{n}{2}-1.$$

References

1. J. J. Sylvester, Mathematical Papers, vol. II, p. 621, Cambridge University Press, 1908.
2. J. S. Frame, this MONTHLY, vol. 46 (1939), p. 304.

ON A PROBLEM OF ERDÖS

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1. Introduction. In what follows, p's denote different primes; p_i denotes the i-th prime, with $p_1=2$; all small letters denote integers $\geqq 0$; m_i denotes an integer in the canonical factorization of which exactly i primes appear to the first power; and M_i denotes the set of all such integers.

Recently, P. Erdös stated and J. Lampek[1] proved that for every k, the equation:

$$\phi(x) = k!, \tag{1}$$

is solvable, ϕ being Euler's totient function.

[1] This MONTHLY, 55, 1948, 103.

The object of this note is to obtain Lampek's solution by an independent method, and to show that the equation has at least one solution in M_1. In fact, if $S_i(k)$ denotes the number of solutions of (1) in M_i, then it is shown that for every k we have

$$S_1(k) \geqq 1, \tag{2}$$

and that, further,

$$S_1(k) \to \infty \quad \text{with} \quad k.$$

These results were conjectured by V. L. Klee,[2] but he did not prove them. Klee was, however, able to prove that

$$S_0(x) = 1. \tag{3}$$

2. Lampek's solution. To get Lampek's solution, we divide integers from 1 to k into two classes A and B. We place $r(\leqq k-1)$ in class A if it is either a prime or one less than a prime, and in class B otherwise. The integer k itself is placed in class A if and only if it is an odd prime. A number of the form $(p-1)$ thus appears in class B first and then shifts (for $k \geqq p$) to class A.

For a given k, let B_k denote the product of integers in class B. We can then write

$$B_k = \prod_{i=1}^{j} p_i^{\alpha_i}, \qquad \alpha_i \geqq 0,$$

where p_j is the greatest prime $\leqq k$.

One solution of (1) is then readily seen to be given by

$$x = p_1^{\alpha_1+1} p_2^{\alpha_2+2} p_3^{\alpha_3+2} \cdots p_j^{\alpha_j+2}$$

$$= \frac{k!(2\cdot 3\cdot 5\cdot 7\cdot \cdots \cdot p_j)}{1\cdot 2\cdot 4\cdot 6\cdot \cdots \cdot (p_j-1)} = \frac{k!}{\prod_1^j \left(1-\frac{1}{p_i}\right)} = \frac{(k!)^2}{\phi(k!)}.$$

This is Lampek's solution.

If $\alpha_1 \geqq 1$, this value of x is in M_0. If $\alpha_1 = 0$, as is the case when $k = 3$, 5 or 7, then

$$x = \frac{(k!)^2}{2\phi(k!)}$$

is another solution of (1); this is in M_0.

It is thus shown that for every k, the equation has a solution in M_0; this can further be shown to be unique.

3. Klee's Conjectures. For values of $k \leqq 8$, Klee's conjecture (2) can easily be

[2] This MONTHLY, 56, 1949, 21.

verified. We may therefore take $k \geqq 9$ in establishing (2).

LEMMA. *If for some $p > k$ we have*

$$(p-1) \mid B_k, \tag{4}$$

then (1) *has a solution in* M_1.

Let

$$\frac{B_k}{p-1} = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot p_3^{\beta_3} \cdots p_j^{\beta_j}, \qquad \beta \geqq 0,$$

where p_j is the greatest prime $\leqq k$; then the required solution is given by

$$x = p_1^{\beta_1+1} p_2^{\beta_2+2} p_3^{\beta_3+2} p_4^{\beta_4+2} \cdots p_j^{\beta_j+2} \cdot p, \quad \text{if} \quad \beta_1 \geqq 1,$$

and by

$$x = p_2^{\beta_2+2} p_3^{\beta_3+2} p_4^{\beta_4+2} \cdots p_j^{\beta_j+2} \cdot p, \quad \text{if} \quad \beta_1 = 0.$$

As interesting examples of (4), we may give the following:

$$72 \mid B_9, \quad 1008 \mid B_{14}, \quad 15120 \mid B_{15}, \quad \text{and} \quad 17136 \mid B_{34},$$

and it follows that Klee's conjecture holds for values of $k \leqq 17136$.

To prove Klee's conjecture, we now give the following result.

THEOREM.[3] *If $k \geqq 32$, and p is any prime $> k$ and $\leqq 2k+1$, then the equation* (1) *has a solution of the form $x = pm_0$.*

Proof. First suppose $(p-1)/2$ is not a prime, and let

$$p - 1 = 2^{\gamma_1} \cdot 3^{\gamma_2} \cdot 5^{\gamma_3} \cdot \cdots \cdot p_t^{\gamma_t}, \qquad \gamma_i \geqq 0,$$

where p_t is the greatest prime $\leqq k/3$.

Evidently then

$$\gamma_1 \leqq 1 + \log_2 k.$$

Since 2^n+1 cannot be a prime except when $n = 2^s$, $s \geqq 0$, it follows that 2^3, 2^5, 2^6, 2^7, 2^9, $\cdots$, are all members of class B. For $k \geqq 32$, the greatest power of 2 which divides B_k is

$$\geqq 3 + 5 + 6 + 7 + 9 + \cdots + \log_2 h,$$

where $h = k/2$ or k according as $\log_2 k$ is or is not of the form 2^s, $s > 0$. This is greater than $1 + \log_2 k$. Hence[3]

$$2^{\gamma_1} \mid B_k \quad \text{when} \quad k \geqq 32.$$

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[3] This holds for $k \geqq 14$, as can be easily verified.

Again, for $\gamma_i > 1$ and $2 \leqq i \leqq t$, $p_i^{\gamma_i}$ belongs to class B. Hence $p_i^{\gamma_i} \mid B_k$, provided $\gamma_i > 1$.

If $\gamma_i = 1$ for any value of i, then since $3p_i$ is a member of class B, p_i still divides B_k.

Hence $(p-1) \mid B_k$, and the theorem follows.

Now, suppose that $(p-1)/2$ is a prime, say p_l, and that[4] $p_l \nmid B_k$. Then $k \geqq p_l > k/3$. Evidently

$$(p-1) \mid p_l(p_l - 1)B_k.$$

Let

$$p_l(p_l - 1)B_k = 2^{c_1} \cdot 3^{c_2} \cdot 5^{c_3} \cdot \cdots \cdot p_l^{c_l} \cdots p_j^{c_j}, \qquad c_i \geqq 0,$$

where p_j is as before the greatest prime $\geqq k$.

Since p_l is not a factor of B_k, we have $c_l = 1$, and we therefore get the solution:

$$x = 2^{c_1} \cdot 3^{c_2+2} \cdot 5^{c_3+2} \cdot \cdots \cdot p_{l-1}^{c_{l-1}+2} \cdot p_{l+1}^{c_{l+1}+2} \cdot \cdots \cdot p_j^{c_j+2} \cdot p.$$

The theorem is thus completely proved.

Klee's second conjecture follows as a corollary, since the number of primes between k and $2k$ goes to infinity with k (as follows easily from the prime number theorem).

CLASSROOM NOTES

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ARTIFICIAL MATHEMATICS

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We teachers of college mathematics take pride in the fact that our subject matter is based on logic, that our derivations and proofs are obtainable by "reasonable" means and are the "natural" consequences of our hypotheses. Yet at times I wonder just how reasonable and natural some of our teaching methods are.

[4] If $p_l \mid B_k$, then $(p-1) \mid B_k$, provided $k \geqq 8$.

How often are "proofs" merely verifications of unmotivated, dogmatic statements? Any experienced teacher who has examined a number of textbooks has, I think, found instances galore. Here I shall cite just three examples of what is unfortunately only too common and which is unfair to the student, unjustified, and unnecessary.

EXAMPLE 1. Consider the linear differential equation of first order,

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

Frequently the method of solving this equation is given in the following form: The function

$$e^{\int P dx} \tag{2}$$

is "seen" to be an integrating factor, for, when we multiply both members of (1) by function (2), we get an exact equation, as may be easily verified.

This *does* produce the result—but where on earth did the function (2) come from? Will not most students ask themselves, if not their instructor, how *they* can be expected to do mathematical thinking if it requires this sort of genius, or at least a crystal ball?

EXAMPLE 2. Consider the linear differential equation of order n with constant coefficients,

$$\phi(D)y = (a_0D^n + a_1D^{n-1} + \cdots + a_n)y = 0. \tag{3}$$

Here a common procedure is to "take" $y=e^{mx}$, so that we get

$$\phi(D)e^{mx} = \phi(m)e^{mx}, \tag{4}$$

which reduces to zero provided that m is a root of the equation $\phi(m)=0$, and therefore we get a solution. But again the student may well ask: how did we hit upon e^{mx} as a function that turns out to do the job so effectively?

EXAMPLE 3. In order to prove the law of the mean for derivatives, the argument sometimes is made to begin, disconcertingly, with: Consider the function

$$g(x) = (a - b)f(x) + [f(b) - f(a)]x + bf(a) - af(b). \tag{5}$$

Since $g(a)=g(b)=0$, and $g'(x)$ must vanish for some value of x between a and b, we get the desired result; but, once more, where did function (5) come from?

These examples, and others of the same ilk, provide illustrations of how artificial mathematical techniques can become. With a little thought and care, however, each proof and derivation can be well motivated, natural, and therefore more acceptable to the student. Moreover, naturalness can then produce the desired confidence in the student, so that he will be more ready to attack new problems and situations himself.

A COLOSSAL PRIMITIVE PYTHAGOREAN TRIANGLE

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In *Scripta Mathematica*, vol. XV, 1949, pp. 247–251 may be found the author's table of non-consecutive positive integral values of 2^n, the highest value of n being 4001. A few of these data enabled him to calculate with great care the lengths of the sides of an enormous primitive right triangle. The numbers of digits in a, b, and c are respectively 1205, 1204, and 1205. The expressions used were $a=2^{4n}+2^{2n+1}$, $b=2^{4n}-2^{4n-2}-2^{2n}-1$ and $c=2^{4n}+2^{4n-2}+2^{2n}+1$, $n=1, 2, 3, \cdots$. (The questioning reader may substitute $u=2^{2n}$ and $v=2^{2n-1}+1$ in equations (8-5) on page 168 of Professor O. Ore's elegant book entitled *Number Theory and its History*.) Putting $n=1000$ we obtain the purely numerical expressions $a=4\cdot 2^{3998}+2^{2001}$, $b=3\cdot 2^{3998}-2^{2000}-1$, and $c=5\cdot 2^{3998}+2^{2000}+1$. The procedure by which the author intentionally adjusted the data so as to cause the first 600-odd figures of a, b, and c to be the same exact multiple of the corresponding elements of the historical triangle 4, 3, 5 should be read clearly from the transformed values

$$a = 2^{3998}(4 + 2^{-1997}) = 2^{3998}[4 + (0.696784785\cdots)10^{-601}],$$

$$b = 2^{3998}(3 - 2^{-1998} - 2^{-3998}) = 2^{3998}[3 - (0.348392392\cdots)10^{-601} - 2^{-3998}],$$

$$c = 2^{3998}(5 + 2^{-1998} + 2^{-3998}) = 2^{3998}[5 + (0.348392392\cdots)10^{-601} + 2^{-3998}].$$

This work is not altogether as fantastic as it may seem to be at first blush. Students just practicing with involution may square the unexpanded numerical values of a, b, c and test their results by the condition $a^2+b^2=c^2$. More proficient pupils may apply their skill to the above functions of n. Obviously many other practice primitive triangles may be found by letting $n=2, 3, 4\cdots$. Later when the student is being introduced to congruences, moduli, residues, etc., he or she may show that the immense constants given below lead to $a\equiv 37084\ (m_5)$, $b\equiv 19983\ (m_5)$, $c\equiv 81659\ (m_5)$, $a^2\equiv 9304\ (m_5)$, $b^2\equiv 16296\ (m_5)$, $a^2+b^2\equiv 25600\ (m_5)$, and $c^2\equiv 25600\ (m_5)$, where m_5 denotes mod. 10^5+1. Then other moduli such as 10^7+1 and $10^{29}+1$ may be tried. The latter is easily applied with a reliable calculating machine and it lends confidence in the present work by giving

$$\begin{aligned}
a &\equiv 515719865\ 6233667385\ 4495330061\ (m_{29})\\
b &\equiv 3686259740\ 4224446729\ 4228735908\ (m_{29})\\
c &\equiv 424331736\ 7521601946\ 0133505593\ (m_{29})\\
a^2 &\equiv 529973252\ 6166485746\ 8892932125\ (m_{29})\\
b^2 &\equiv 648590322\ 1905618795\ 2838387251\ (m_{29})\\
c^2 \equiv a^2 + b^2 &\equiv 178563574\ 8072104542\ 1731319375\ (m_{29}).
\end{aligned}$$

Finally the functioning of an electronic calculating machine may be verified by actually squaring the 1204 and 1205-digit numbers tabulated on page 332.

$a =$ 13182

0409343094	3100103889	7942365913	6318401916	1093272769	0928034502
4175692811	2834455107	9752123172	1220331409	4075648071	6823038446
8176942405	8128173106	2452512184	0385446744	4438688895	6328970642
7719939300	3658655292	4249514488	8321833894	1583237562	0009284922
6089461110	3857875407	7913265440	9185831255	8605043164	7284603636
4908238500	0782681167	2468900210	6891044880	8948534719	2152708820
1197650061	2594485839	7761874669	3012787452	3350479658	6994514054
4352170538	0373270324	0283400815	9261693483	6479947271	6094576894
0072431686	6256888660	3065832486	8306061250	1764335646	9732407252
8745672177	3369482423	6675323341	7556818392	2195469382	0456072249
8800234260	7773169142	5276429749	2719440301	2786312910	8110276523
1791250003	9324450202	0158585930	4304919378	1774876967	1953220419
0542848312	1124562345	8850343351	5183070273	3949122803	8974497727
1904776640	9658550797	9096195582	3614866289	5884470059	3540289002
5614727146	4790864241	1262007050	8297458635	8000281716	8946915944
4567759251	1693734052	7736385060	5142174692	5088960985	2079084482
3799705399	8732549654	4541159162	4129752556	5023893091	9749021654
2178578874	6307774977	0114507233	6119847609	9196512050	4342804980
8328964656	9380080473	2217857144	6992977330	4159099451	1618533227
2629192527	9641335050	2194424741	0970572936	3002894540	9209008128

$b =$ 9886

5307007320	7325077917	3456774435	2238801437	0819954576	8196025876
8131769608	4625841330	9814092379	0915248557	0556736053	7617278835
1132706804	3596129829	6839384138	0289085058	3329016671	7246727982
0789954475	2743991469	3187135866	6241375420	6187428171	5006963691
9567095832	7893406555	8434949080	6889373441	8953782373	5463452727
3681178875	0587010875	4351675158	0168283660	6711401039	4114531615
0898237545	9445864379	8321406001	9759590589	2512859744	0245885540
8264127903	5279952743	0212550611	9446270112	7359960453	7070932670
5054323764	9692666495	2299374365	1229545937	6323251735	2299305439
6559254133	0027111817	7506492506	3167613794	1646602036	5342053900
3773437509	9466771036	0657027891	4579524431	7037517303	0963296708
1779021849	4684079805	0954067868	6963014737	5182041750	7099221952
5109734076	4460136111	1298715850	3423824455	3175514032	9229151347
5222581358	9386681350	9163431074	8170763126	0316261557	6862651650
7552196087	2344008640	2449482926	5482953394	0587827246	6155261126
6392215215	6524185913	1796340083	3658779546	8583026613	7986806570
0184133286	8547379181	7702639968	6400231487	2118170969	8766448686
2137474054	9190867936	9457028201	6242333409	4258644591	9282923637
7244674562	2877414666	0712328899	2094344789	0922719730	3577627632
2577297330	3437142405	4643761448	8957480565	8811624692	9034182655

$c =$ 16477

5511678867	8875129862	2427957392	0398002395	1366590961	3660043128
0219616014	1043068884	9690153965	1525414261	7594560089	6028798058
5221178007	2660216382	8065640230	0481808430	5548361119	5411213303
4649924125	4573319115	5311893111	0402292367	6979046952	5011606153
2611826387	9822344259	7391581801	1482289069	8256303955	9105754545
6135298125	0978351459	0586125263	3613806101	1185668399	0190886025
1497062576	5743107299	7202343336	6265984315	4188099573	3743142568
0440213172	5466587905	0354251019	9077116854	5599934089	5118221117
5090539608	2821110825	3832290608	5382576562	7205419558	7165509066
0932090221	6711853029	5844154177	1946022990	2744336727	5570090140
1304249914	5898597935	6615360534	2923266900	0451560710	5066199243
4500413112	8270008044	6699309465	5621744345	0529126624	1822109506
7500119095	5575731543	3859504191	4200750891	9064606662	8717088990
2657370128	1358849448	8775015110	7794350907	2897332982	0949822192
5023099369	5239096577	0479295396	5927738945	0752921721	4850689431
5489536530	3269498790	1266912098	4309007482	1220984756	4455351527
9150244292	0114467232	6254511311	1143940937	5290017055	5059086535
1025347532	6560740743	7765822707	8641278133	5912396395	5043998167
3009976462	9230513178	2201683055	2850988737	7880911399	1441403161
1649732420	5775353483	4541796662	2150946688	5689290448	4787716097

PROBLEMS ON NATURAL NUMBERS

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In a recent issue of this journal (see *On the System of Natural Numbers*, M. O. Gonzalez and J. D. Mancill, this Monthly, vol. 57 (1950), pp. 104–111. All references will be to this paper unless stated otherwise) there appeared a treatment of natural numbers intended to stimulate logical thinking on the part of students of college algebra. As further stimulation, I have compiled a set of home work problems for the students based on this treatment. Answers are given after the problems.

1. Prove that there is a set with a negative number of elements.
2. How would one prove the existence of a set with exactly one element?
3. Prove that if B precedes A, then A is the immediate predecessor of B.
4. Prove that $b \times 1$ is defined by the first sentence of Section 11 and does not need further definition.
5. What step was omitted in the proof of the Principle of Complete Induction?
6. State why the proof of the theorem at the top of p. 108 is incorrect, and give a correct proof.
7. Prove that Postulate 3 is not the Axiom of Zermelo.
8. Prove that the conditions of Postulate 4 do not characterize "linear order."

SOLUTIONS

1. Start with a set containing a single element. By Postulate 2 we can remove an element and get a set with no elements. By Postulate 2 again, we can remove another element and get a set with one less than no elements.

2. One wouldn't. All postulates concerning construction of sets start from given sets, but no postulate gives a set to start with. We cannot use Definition 1 since it gives us no authority to construct a set, but merely tells how to decide which sets are determined or defined. Indeed all postulates and definitions are consistent with the supposition that there are no sets at all, and so cannot possibly prove the existence of a set.

3. If B precedes A, then by Postulate 4 there does not exist an element C such that A precedes C and C precedes B. So by Definition 5, A is the immediate predecessor of B.

4. In the definition of sum at the bottom of p. 111, there is no stipulation as to the number of sets $A, B, C, \cdots, N$. By taking this number to be unity, we have a definition of sum for the case of a single summand. Hence $b \times 1$ is defined.

5. By (2) of Definition 8, either K is the first element or K has an immediate predecessor. Treatment of the case where K is the first element was omitted.

6. The Principle of Complete Induction provides for proofs of properties of elements of sets, and the theorem in question states a property of the entire set. Prove by induction on H the Lemma:

That portion of the set

$$X = A, B, C, \cdots, H, I, \cdots, L$$

consisting of H and all preceding elements can be made empty by removing its elements one at a time.

This Lemma states a property of the element H, and so can be proved by the Principle of Complete Induction, using essentially the reasoning given at the top of p. 108.

7. The Axiom of Zermelo (see *Untersuchungen über die Grundlagen der Mengenlehre* I, E. Zermelo, *Math. Annalen*, vol. 65 (1908), pp. 261–281) states that if one has given an infinite number of non-empty, non-overlapping sets, one can choose a unique element from each. The discrepancy with Postulate 3 is apparent.

8. The conditions of Postulate 4 characterize "partial order" (see p. 1 of *Lattice Theory* by G. Birkhoff, Revised Edition, Am. Math. Soc., 1948). For "linear order," an additional condition, "If A and B are distinct then either A precedes B or B precedes A," is needed (see p. 10 of *Lattice Theory*).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 916. *Proposed by H. D. Larsen, Albion College*

As a variation of the popular game of *Twenty Questions*, suppose I think of a number which you are to determine by asking me not more than twenty questions, each of which can be answered by only "yes" or "no." What is the largest number that I should be permitted to choose that you may determine it in twenty questions?

E 917. *Proposed by Lawrence Ringenberg, Eastern Illinois State College*

Find all solutions of the equation $1/x^2+1/y^2=1/z^2$, where x, y, z are relatively prime positive integers.

E 918. *Proposed by N. A. Court, University of Oklahoma*

Two spheres are orthogonal if they cut a given line in pairs of harmonic points and if the two planes determined by that line and the centers of the two spheres are perpendicular.

E 919. *Proposed by Leo Moser, University of Manitoba*

Show that the greatest common divisor of a and b is given by

$$(a, b) = \sum_{m=0}^{a-1} \sum_{n=0}^{a-1} (1/a)e^{2\pi i b mn/a}.$$

E 920. *Proposed by T. G. Room, University of Sydney, Australia*

If S and T are any two square matrices of the same order, and the necessary matrices are non-singular, then

$$(I + S)^{-1}(S + T)(I + ST)^{-1}(I + S) = (I - S)(I + TS)^{-1}(S + T)(I - S)^{-1}.$$

SOLUTIONS

Minimal n Partitionable into Two Squares in Given Number of Ways

E 883 [1949, 552]. *Proposed by H. T. R. Aude, Colgate University.*

Find the smallest integer n which is the sum of the squares of two non-negative integers in exactly (a) three ways, (b) five ways, (c) six ways.

Solution by C. M. Sandwick, Easton High School, Easton, Pa. Let $P_2(n)$ denote the number of partitions of n into two non-negative integral squares. Then our problem is to find the minimum values of n that satisfy the equations $P_2(n)=3$, $P_2(n)=5$, $P_2(n)=6$. If $P_2(n)>1$, then $n=2^rMQ^2$, where $r\geqq 0$, M is the product of two or more prime factors each of the form $4x+1$, and Q contains no prime factors of this form. The value of $P_2(n)$ is equal to the number of distinct ways in which M can be represented as the product of two positive integers.

If A, B, C are unequal primes of the form $4x+1$, it is easy to show that $P_2(n)=3$ only when M has one of the forms A^2B, A^4, A^5; $P_2(n)=5$ only when M has one of the forms A^2B^2, A^4B, A^8, A^9; and $P_2(n)=6$ only when M has one of the forms A^2BC, A^3B^2, A^5B, A^{10}, A^{11}. If $C<AB$ and $B<A^2$, the first named form of M in each category yields the smallest value. These conditions are met when A, B, C take the values 5, 13, 17, respectively, the smallest primes of the form $4x+1$. Therefore the smallest values of n for which $P_2(n)=3$, 5, 6, respectively, are (a) $(5^2)(13)=325$, (b) $(5^2)(13^2)=4225$, (c) $(5^2)(13)(17)=5525$. In fact we have

$$325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2,$$

$$4225 = 0^2 + 65^2 = 16^2 + 63^2 = 25^2 + 60^2 = 33^2 + 56^2 = 39^2 + 52^2,$$

$$5525 = 7^2 + 74^2 = 14^2 + 73^2 = 22^2 + 71^2 = 25^2 + 70^2 = 41^2 + 62^2 = 50^2 + 55^2.$$

Also solved by B. H. Brown, W. E. Buker, Roger Lessard, C. S. Ogilvy, and the proposer.

Editorial Note. It may similarly be shown that $n=(5)(13)(16)=1105$ is the minimum n such that $P_2(n)=4$. If we decide to omit zero squares, then $n=(5^4)(13)=8125$ is the minimum n such that $P_2(n)=5$. The number of partitions of an integer into two squares is given in detail in Uspensky and Heaslet, *Elementary Number Theory*, pp. 336–42.

Concurrent Cevians of a Tetrahedron

E 885 [1949, 552]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 divide a set of concurrent cevians AA', BB', CC', DD' in the same ratio, and let A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Show that triangles A_2CD, A_3DB, A_4BC have equal areas.

I. *Modification by the Proposer.* It has been found that the problem as originally stated is incorrect and we shall here prove the following modification:

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 be points dividing the cevians AA', BB', CC', DD', through a point Q in the same ratio $AA_1/A_1A' = \cdots = k$, and let the lines A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Then triangles A_2CD, A_3DB, A_4BC have equal areas when (1) *Q is arbitrary and $k=1$,* (2) *k is arbitrary and Q coincides with the centroid of $ABCD$.*

If we designate the barycentric coordinates of Q by (a, b, c, d), those of A_1, B_1, C_1 are of the form

$$A_1\colon (a', b, c, d), \qquad B_1\colon (a, b', c, d), \qquad C_1\colon (a, b, c', d),$$

where

$$a' = (b + c + d)/k, \qquad b' = (c + d + a)/k, \qquad c' = (d + a + b)/k.$$

On A_1B_1 one has the coordinates

$$a' + \lambda a, \qquad b + \lambda b', \qquad c(1 + \lambda), \qquad d(1 + \lambda),$$

whence those of A_2 are

$$0, \qquad ab - a'b', \qquad c(a - a'), \qquad d(a - a'),$$

and we have

$$A_2CD = BCD(ab - a'b')/[ab - a'b' + (a - a')(c + d)].$$

Similarly

$$A_3DB = BCD(ac - a'c')/[ac - a'c' + (a - a')(b + d)].$$

Consequently, the condition $A_2CD = A_3DB$ is equivalent to

$$(ab - a'b')(b + d) = (ac - a'c')(c + d),$$

or

$$a(b-c)(b+c+d) = a'[b'(b+d) - c'(c+d)],$$

or, substituting the values of a', b', c',

$$a(b-c)(b+c+d) = (b+c+d)(b-c)a/k^2,$$

or, finally,

$$a(b-c)(b+c+d)(1-k^2) = 0.$$

This equality leads directly to the two announced cases, and the modified problem is established.

II. *Modification by Joseph Langr, Prague, Czechoslovakia.* Although the original problem is generally not true the following slight modification leads to a correct version.

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 be points on the cevians AA', BB', CC', DD' through a point Q such that

$$A'A/AA_1 = BB'/B'B_1 = CC'/C'C_1 = DD'/D'D_1 = u,$$

and let A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Then triangles A_2CD, A_3DB, A_4BC have equal areas.

Let the plane of the lines AA', BB' cut the edge CD in B''. Through A_1 draw the parallel to BB'' to cut AB in P and through B_1 the parallel to AB'' to cut BB'' in N and AB, of course, in P. Then

$$PA_1 = BA'/u, \qquad NB'' = BB''/u.$$

From the similar triangles A_1B_1P and A_2B_1N we have

$$A_2N = PA_1(B_1N/PB_1) = (BA'/u)(B'B''/AB'),$$

and consequently

$$A_2B'' = A_2N + NB'' = (BA'/u)(B'B''/AB') + BB''/u,$$

and

$$A_2B''/BB'' = (1/u)[1 + (B'B''/AB')(BA'/BB'')].$$

But, by the theorem of Menelaus applied to triangle $AA'B''$ and the transversal BB',

$$(B'B''/AB')(BA'/BB'') = AQ/QA',$$

whence

$$A_2B''/BB'' = (1/u)[1 + AQ/QA'] = v,$$

say. Therefore

$$A_2CD/BCD = A_2B''/BB'' = v.$$

Similarly

$$A_3DB/BCD = A_4BC/BCD = v,$$

and the modified problem is established.

Editorial Note. By considering the last equation in the solution to the proposer's modified problem, along with the two similar equations gotten by cyclically permuting b, c, d, we see that triangles A_2CD, A_3DB, A_4BC have equal finite areas in the following three cases: (1) Q arbitrary and $k=1$, (2) k arbitrary and Q on the line determined by A and the centroid of BCD, (3) k arbitrary and Q on the plane of BCD

If in Langr's modification, we take $u=2$, we obtain the first case of the proposer's modification, and this is the space analogue of the first part of E 879 [1949, 474]. Space analogues of the other two parts of E 879 may also be established.

A Circle and Variable Chord

E 886 [1949, 632]. *Proposed by P. L. Chessin and R. Gilman, New York City*

Let M be a fixed point on the radius OR of a circle of center O and set $OR/OM=k$. If AB is a variable chord such that $\sphericalangle AMR=\sphericalangle OMB$, then (1) AB passes through a fixed point on line OR, (2) $AB=k(MB-MA)$, (3) A, B, O, M are concyclic.

Solution by Roscoe Woods, State University of Iowa. Let MR and the perpendicular to OR at M cut AB in N and C respectively. Then MC bisects angle AMB, and $(NCAB)$ is harmonic. Thus N is the pole of line MC, and is therefore fixed. This establishes (1).

It is now obvious that the inverse of AB with respect to the given circle is a circle passing through A, B, M, and O. This demonstrates (3).

If the theorem of Ptolemy be employed with reference to the inscribed quadrilateral $ABOM$, we have $(OM)(AB)+r(MA)=r(MB)$, where r is the radius of the given circle. This relation reduces to (2) when use is made of the given condition $OR/OM=k$.

Also solved by Ragnar Dybvik, P. W. M. John, L. M. Kelly, Roger Lessard, C. S. Ogilvy, Nathan Schwid, A. Sisk, M. C. Stapp, and the proposers.

An Impossible Digit Relation

E 887 [1949, 632]. *Proposed by C. W. Trigg, Los Angeles City College*

Let N be an n-digit integer and M the kn-digit integer formed by writing N down k times. Show that there is no scale of notation in which $N^k=M$ if $k>1$.

Solution by N. G. Gunderson, University of Rochester. Let N be written to the base r, so that $0<N<r^n$, and suppose that $N^k=M=N\Sigma_{i=0}^{k-1} r^{ni}$ for some $k>1$. But this would imply $\Sigma_{i=0}^{k-1} r^{ni}=N^{k-1}<r^{n(k-1)}$, which is impossible.

Also solved by N. D. Lane, Roger Lessard, C. S. Ogilvy, C. M. Sandwick, and the proposer.

Passing a Cube through a Cube of Same Size

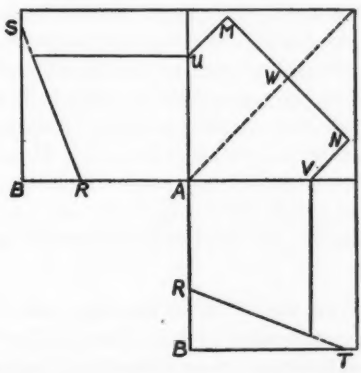
E 888 [1949, 632]. *Proposed by H. D. Grossman, New York City*

Show how to cut a hole in a cube through which another cube of equal size can pass.

I. *Solution by C. S. Ogilvy, Columbia University.* The projection of a unit cube on a plane perpendicular to a main diagonal is a regular hexagon whose inscribed circle has diameter $\sqrt{2}$. Hence any unit square hole (prism) whose axis is this main diagonal has its cross section circumscribed by the circle. Therefore such a hole is completely surrounded by material of the cube, provided only that it is not oriented with a diagonal of its cross section coinciding with a line joining the midpoints of two opposite edges of the cube.

II. *Solution by F. Bagemihl, University of Rochester.* Let the vertices of the given cube have rectangular coordinates (0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1), (0, 1, 1). The points A (0, .9, 1), B (.9, 0, 1), C (1, .1, 0), D (.1, 1, 0) are on the cube, and are the vertices of a rectangle with $AB=CD=\sqrt{1.62}>1$, $AD=BC=\sqrt{1.02}>1$, so that a square, s, can be drawn wholly interior to the rectangle $ABCD$, each side of the square having a length greater than unity. Through the given cube cut a hole with a square cross section, perpendicular to $ABCD$, and intersecting the latter in s. A unit cube can pass through this hole if four faces of the cube are parallel, respectively, to the four faces of the hole.

Also solved by R. V. Andree, Sholom Arzt, Sidney Kravitz, Mary Ann Lee, Roger Lessard, W. R. Ransom, L. A. Ringenberg, C. M. Sandwick, O. D. Smith, G. W. Walker, and D. G. Williams.



Williams furnished diagrams, based upon the symmetrical orientation in solution I, showing how to mark the intersections of an appropriate hole with the faces of the given cube. The essential information of his diagrams is contained in the accompanying figure, representing three faces of a trihedral angle of the

cube. By symmetry considerations the other three faces may be similarly marked. If e is the edge of the cube, then

$$AW = e\sqrt{3}/2, \qquad MW = WN = e/2,$$
$$AR = e\sqrt{6}/4, \qquad BS = BT = e(4 - \sqrt{6})/2.$$

Comparison of Some Solid Angles

E 889 [1949, 632]. *Proposed by Michael Golomb, Purdue University*

Let k be a circle with its interior, F_1F_2 one of its diameters, P a point on the axis of the circle, F_1PF_2 a circular arc, and A_1PA_2 a semiellipse with foci at F_1, F_2. Show that the solid angles subtended by k at the points of arc F_1PF_2 are no smaller, and at the points of arc A_1PA_2 are no greater, than the solid angle subtended by k at P.

Solution by the Proposer. If Q is any point in the plane F_1PF_2 and w is the bisector of $\sphericalangle F_1QF_2$, then a plane perpendicular to w cuts the cone $Q(k)$ in an ellipse whose center is on w, whose minor axis is in the plane F_1PF_2, and whose major axis is parallel to the plane of k.

Now let Q_1 be a point on arc F_1PF_2, w_1 the bisector of $\sphericalangle F_1Q_1F_2$, and O_1 the point on w_1 such that $Q_1O_1 = PO$. Then, since $\sphericalangle F_1PF_2 = \sphericalangle F_1Q_1F_2$, the elliptic cross section of the cone $Q_1(k)$ with center at O_1 and perpendicular to w_1 has a minor axis equal to the diameter of k. Hence the solid angle subtended by k at Q_1 is no smaller than that at P.

Next, let Q_2 be a point on arc A_1PA_2, w_2 the bisector of $\sphericalangle F_1Q_2F_2$, and O_2 the point of intersection of w_2 with F_1F_2. The major axis of the elliptic cross section of the cone $Q_2(k)$ with center O_2 and perpendicular to w_2 lies in the plane of k, and therefore is the chord $M_1O_2M_2$ of k which is perpendicular to F_1F_2. Since w_2 is the bisector of $\sphericalangle F_1Q_2F_2$, w_2 is normal to the semiellipse A_1PA_2. It is a general property of an ellipse that the ratio of the length of the normal Q_2O_2 to that of the chord M_1M_2 in the circle through the foci is independent of the position of Q_2 on the ellipse. Therefore $Q_2O_2/M_1M_2 = PO/F_1F_2$. Hence an elliptic cross section of the cone $Q_2(k)$ perpendicular to w_2 and at a distance from Q_2 equal to PO has a major axis equal to $(M_1M_2)(PO)/(Q_2O_2) = F_1F_2$, *i.e.*, equal to the diameter of the circle k. Consequently, the solid angle subtended by k at Q_2 is no greater than that at P.

Editorial Note. Let Σ be the locus, on the same side of k as P, of a point Q such that the solid angle subtended by k at Q is equal to that subtended by k at P. The above solution establishes, in an elementary fashion, that the surface of revolution Σ lies between the spherical cap obtained by rotating arc F_1PF_2 about PO, and the oblate hemispheroid obtained by rotating arc A_1PA_2 about PO.

A Composition of a Set of Integers

E 890 [1949, 632]. *Proposed by Leo Moser, University of Manitoba*

Let $a_1, \cdots, a_n$ be n integers with $a_1 = 1$ and $a_i \leqq a_{i+1} \leqq 2a_i$, $i = 1, \cdots, n-1$. Show that there exists a sequence $\{\epsilon_i\}$ of plus and minus ones such that $\Sigma_{i=1}^{n} \epsilon_i a_i = 0$ or 1.

Solution by S. H. Gould, Purdue University. If $f(k)$ denotes the smallest non-negative integer representable in the form $\Sigma_{i=n-k}^{n} \epsilon_i a_i$, then $f(0) = a_n$, and if we assume $f(k) \leqq a_{n-k}$, then

$$f(k+1) \leqq |f(k) - a_{n-k-1}| \leqq a_{n-k-1},$$

so that $f(n-1) \leqq a_1$

Also solved by A. L. Epstein, Paul Erdös, N. J. Fine, A. W. Fuller, N. G. Gunderson, Roger Lessard, D. J. Newman, A. Rosenthal, and the proposer.

Newman showed that, under the given hypothesis, one can find a sequence $\{\epsilon_i\}$ of plus and minus ones such that $\Sigma_{i=1}^{n} \epsilon_i a_i = k$ or $k+1$, where k is any integer such that $|k| < 2a_n$.

Erdös established the following more general theorem. "Let $0 < a_1 < \cdots < a_n$, be a sequence of real numbers such that $a_{i+1} \leqq 2a_i$, $i = 1, \cdots, n-1$. Then every closed interval (u, v) of length $2a_1$ satisfying

$$-2a_1 - \sum_{i=1}^{n} a_i \leqq u < v \leqq 2a_1 + \sum_{i=1}^{n} a_i$$

contains a number of the form $\Sigma_{k=1}^{n} \epsilon_k a_k$, $\epsilon_k = \pm 1$."

The proposer said that the problem was suggested by the following statement in Dickson, History of the Theory of Numbers, vol. 1, p. 436. "H. F. Scherk stated (1833) the empirical theorem: Every prime of odd rank can be composed by the addition and subtraction of all smaller primes, each taken once, thus $13 = 1+2-3-5+7+11$." This now follows easily from Bertrand's postulate (the existence of a prime between n and $2n$) and the result of the problem.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4395. *Proposed by David Gale, Princeton University*

Let S be a set of vectors (points) in n-space with the property that every vector in n-space can be expressed as a linear combination of vectors from S with non-negative coefficients. Show that one can find a subset R of S containing not more than $2n$ elements such that R has this same property.

4396. *Proposed by D. H. Browne, Buffalo, New York*

With the notation

$$P(s, t) = \prod_{m=0}^{s} (1 + mt),$$

let

$$S(n, t) = P(n, t) \sum_{k=0}^{n} k!t^k/P(k, t).$$

Show that for $n \geqq 1$

$$S(n, t) \equiv P(n - 1, t) \pmod{n + 1},$$

and that for $n > 2$, $t \leqq n$,

$$S(n, t) \equiv 0 \pmod{t + 1}.$$

4397. *Proposed by Albert Wilansky, Lehigh University*

Suppose $f''(0)$ exists, is finite, and is different from 0. Assume also $f(0) = 0$. For sufficiently small $h > 0$ we may write

$$\frac{f(-h)}{h} = f'(\xi), \qquad \frac{f(h)}{h} = f'(\zeta),$$

with $-h < \xi < 0 < \zeta < h$. Prove that

$$\lim_{h \to 0} \frac{\zeta - \xi}{h} = 1.$$

4398. *Proposed by Lucien Droussent, Clermont Ferrand, France*

The Lemoine point of a rectangular hyperbola for an inscribed triangle T lies on the orthic axis of T, and the polar of this point with respect to the hyperbola is tangent to the conic which touches the sides of T at the feet of its altitudes and has the symmedian point of T for center. (The tangents to a conic at the vertices of an inscribed triangle T form a triangle which is perspective to T. The center of perspectivity is called the *Lemoine point* of the conic for the triangle T.)

4399. *Proposed by Ky Fan, University of Notre Dame*

Let $\{f(n)\}$ and $\{g(n)\}$ be two sequences of natural numbers defined by the following three conditions:

(1) $f(1)=1$.

(2) $g(n)=na-1-f(n)$, a being an integer >4.

(3) $f(n+1)$ is the least natural number distinct from the $2n$ numbers $f(1)$, $f(2), \cdots, f(n)$; $g(1), g(2), \cdots, g(n)$.

Prove that there exist constants α and β such that

$$f(n) = [\alpha n], \qquad g(n) = [\beta n].$$

The brackets denote, as usual, the greatest integer function.

SOLUTIONS

Morris' Lottery

4312 [1948, 505]. *Proposed by R. C. Lyness, Staffordshire, England*

In order to help our school savings campaign, Morris organized a lottery. Certain members of the savings group were each issued a book containing a gross of tickets. The tickets were sold for a penny each and the prize for the winning ticket was a number of six-penny stamps. When I asked Morris how many stamps the winner got, he said I could work it out for myself. All he need tell me was that before the winning ticket was drawn all the ticket sellers had assembled around a circular table and each had handed in his takings and his unsold tickets; that (a) no two had sold the same number: that (b) the number of unsold tickets returned by each seller was in every case equal to the product of the values in shillings of the tickets sold by his two neighbors at the table; and that (c) when the ticket sellers were rewarded with a penny each for their trouble the sum left over exactly bought the winner's stamps.

How many sellers were there and how many tickets did each sell?

Solution by N. J. Fine, University of Pennsylvania. Let $A_1, A_2, \cdots$, be the amounts, in pence, sold by the n sellers. Extend the sequence indefinitely, so that $A_{n+k}=A_k$. By condition (a), $A_j=A_k$ implies $j\equiv k \pmod{n}$. It is easily seen that $n\neq 1, 2$, and that

$$0 < A_k < 144. \tag{1}$$

Condition (b) may be written

$$144 - A_k = A_{k-1}A_{k+1}/144. \tag{2}$$

Given A_1, A_2, we compute A_3, A_4, A_5, A_6, from (2), and find that $A_6 = A_1$. Hence $n = 5$.

Now no A_k is prime to 144, for then A_{k+2} is divisible by 144, which contradicts (1). Furthermore, since $\sum_{k=1}^{5} A_k \equiv 5 \pmod 6$ by (c), at least one of the A_k, say A_1, is odd and therefore divisible by 3. Applying (2) for $k = 2, 5$, it is clear that both A_3 and A_4 are divisible by 16. Similarly, if A_2 were odd 16 would divide A_5, hence also

$$144 - A_5 = A_1A_4/144; \tag{3}$$

hence also $A_4/16$, since A_1 is odd, so that $A_4 \geqq 256$ contradicting (1). Therefore A_2 and A_5 (by similar argument) are both even. Using (3) again, $A_4/16$ is even, and so is $A_3/16$. Thus we have

$$A_1 = 3B_1, \quad A_2 = 2B_2, \quad A_3 = 32B_3, \quad A_4 = 32B_4, \quad A_5 = 2B_5,$$

with the additional conditions that B_1 is odd and that

$$1 \leqq B_3 \leqq 4, \quad 1 \leqq B_4 \leqq 4, \quad B_3 \neq B_4. \tag{4}$$

From (4), neither A_3 nor A_4 is divisible by 9; so from (2), with $k = 3, 4$, we find that both A_2 and A_5 are divisible by 3, hence by 6. Therefore $\sum_{k=1}^{5} A_k \equiv 3B_1 + 2B_3 + 2B_4 \equiv 5 \pmod 6$,

$$B_3 + B_4 \equiv 1 \pmod 3. \tag{5}$$

Also

$$\begin{aligned} 144 - A_3 &= A_4(A_2/144) < A_4, \\ 32(B_3 + B_4) &= A_3 + A_4 > 144, \\ B_3 + B_4 &> 9/2, \\ B_3 + B_4 &\geqq 5. \end{aligned} \tag{6}$$

The only combination of values satisfying (4), (5), (6) is 3 and 4. We may therefore take $A_3 = 96$, $A_4 = 128$, and from these by (2) obtain the complete solution

$$A_1 = 135, \quad A_2 = 54, \quad A_3 = 96, \quad A_4 = 128, \quad A_5 = 24.$$

(Interchanging the values of A_3 and A_4 gives the same set in reverse order.)

Also solved by the Proposer, whose major interest is in the illustration of the 5-cycle. He remarks that the recurrences, $u_{n+1}u_n = 1$, $u_{n+2}u_n = u_{n+1} + 1$, $u_{n+3}u_n = u_{n+2} + u_{n+1} + 1$ give respectively a 2-cycle, 5-cycle, and 8-cycle. The recurrence of the present problem is obtained by putting $u_k = -A_k/144$.

Subset of a Group

4317 [1948, 586]. *Proposed by Leo Moser, University of Manitoba*

Let G be an Abelian group and A a subset of order n, such that $a \in A$ implies $a^{-1} \in G - A$. Consider the n^2 (not necessarily distinct) elements of G of the form $a_i a_j$, a_i and a_j elements of A. Show that of these n^2 elements at most $\binom{n}{2}$ are elements of A.

Solution by Robert Steinberg, University of California, Los Angeles. For each $a_i a_j = a_k$ in A there is an $a_k a_j^{-1} = a_i$ in A, whence the number of elements $a_i a_j$ in A is the same as the number of elements $a_i a_j^{-1}$ in A. Consider the n^2 elements $a_i a_j^{-1}$. From the conditions on A, we see that $a_i a_i^{-1} = I$ is not in A and that $a_j a_i^{-1} = (a_i a_j^{-1})^{-1}$ is not in A if $a_i a_j^{-1}$ is in A. Thus at most $\frac{1}{2}(n^2 - n) = \binom{n}{2}$ elements $a_i a_j^{-1}$ are in A, and hence at most $\binom{n}{2}$ elements $a_i a_j$ are in A.

Also solved by Olga Taussky and John Todd, and by the Proposer.

Editorial Note. That $\binom{n}{2}$ is the best possible value is easily seen on consideration of a cyclic group G. The above proof shows also that the hypothesis that G be Abelian is not necessary.

Infinite Series

4318 [1948, 586]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that

$$\frac{1}{(\sinh \pi)^2} + \frac{1}{(2 \sinh 2\pi)^2} + \frac{1}{(3 \sinh 3\pi)^2} + \cdots = \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right) - \frac{11\pi^2}{180}.$$

Solution by the Proposer. Take

$$\frac{\pi}{x} \coth \pi x = \frac{1}{x^2} + \frac{2}{x^2 + 1^2} + \frac{2}{x^2 + 2^2} + \cdots,$$

differentiate with respect to x and divide across by $-x$ to obtain

$$\frac{\pi}{x^3} \coth \pi x + \frac{\pi^2}{x^2 \sinh^2 \pi x} = \frac{2}{x^4} + \frac{4}{(x^2 + 1^2)^2} + \frac{4}{(x^2 + 2^2)^2} + \cdots,$$

and thence

$$\pi \sum_1^\infty \frac{\coth \pi n}{n^3} + \pi^2 \sum_1^\infty \frac{1}{(n \sinh \pi n)^2} = 2\zeta(4) + 4 \sum_1^\infty \sum_1^\infty \frac{1}{(m^2 + n^2)^2}.$$

Now

$$\sum_1^\infty \sum_1^\infty \frac{1}{(m^2 + n^2)^s} = \zeta(s)\left(\frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \cdots\right) - \zeta(2s),$$

and it remains to evaluate

$$\pi\sum_1^\infty \frac{\coth \pi n}{n^3} = \sum_1^\infty \frac{1}{n^4} + 2\sum_1^\infty \sum_1^\infty \frac{1}{n^2(n^2+m^2)}$$

$$= \sum \frac{1}{n^4} + 2\sum \sum \frac{1}{m^2}\left(\frac{1}{n^2} - \frac{1}{m^2+n^2}\right)$$

$$= \sum \frac{1}{n^4} + 2\sum \frac{1}{m^2}\sum \frac{1}{n^2} - 2\sum \sum \frac{1}{m^2(m^2+n^2)}$$

$$= \sum \frac{1}{n^4} + 2\sum \frac{1}{m^2}\sum \frac{1}{n^2} + \sum \frac{1}{m^4} - \pi\sum \frac{\coth \pi m}{m^3},$$

and hence

$$\pi\sum_1^\infty \frac{\coth \pi n}{n^3} = \frac{\pi^4}{90} + \frac{\pi^4}{36} = \frac{7\pi^4}{180}.$$

The desired result follows immediately upon substitution.

Exceptions to the Converse of Fermat's Theorem

4319 [1948, 586]. *Proposed by Paul Erdös, Syracuse University*

If p is a prime greater than 3 and if $n=(2^{2p}-1)/3$, show that 2^n-2 is divisible by n.

Solution by J. E. Wilkins, Jr., American Optical Company, Buffalo, New York. In

$$n - 1 = 4(2^{p-1} + 1)(2^{p-1} - 1)/3$$

the factor 4 is divisible by 2, the factor $2^{p-1}-1$ is divisible by p (Fermat's simple theorem, p being prime) and by 3 (since p is odd), hence by $3p$ when p is greater than 3. Therefore $n-1$ is divisible by $2p$.

By hypothesis $2^{2p}-1$ is divisible by n. Since $2p$ is a divisor of $n-1$ it follows that $2^{n-1}-1$ is divisible by n and we have the desired conclusion.

Also solved by Daniel Block, R. H. Breusch, A. W. Boldyreff, John Kelly, Leo Moser, Alex Rosenberg, N. C. Scholomiti, J. D. Swift, and the Proposer.

Editorial Note. For all $p>3$, the Proposer notes that n is composite and thus we have infinitely many exceptions to the converse of Fermat's theorem. Sierpinski proved (*Colloquium Math.* 1, 9) that if n is composite and divides 2^n-2, then 2^n-1 has the same property. Note also that the original proposal is true for $p=2$, although then n is not composite.

See also *On the Converse of Fermat's Theorem* by the Proposer, This MONTHLY, vol. 56 (1949) pp. 623–624.

Series with Irrational Sum

4321 [1948, 642]. *Proposed by Paul Erdös, Syracuse University*

Let $n_1<n_2<\cdots<n_k<\cdots$ be a sequence of integers such that

$$\lim_{k\to\infty}\frac{n_k}{n_1n_2\cdots n_{k-1}}=\infty.$$

Show that $\sum_{i=1}^{\infty}1/n_i$ is irrational.

Solution by Robert Steinberg, University of California, Los Angeles. Suppose, if possible, that $\sum_{i=1}^{\infty}1/n_i=p/q$, in which p and q are integers. From the hypothesis we can choose k such that

$$\frac{n_k}{n_1n_2\cdots n_{k-1}q}>3$$

and $n_{i+1}>3n_i$ for $i\geqq k$. Then

$$pn_1n_2\cdots n_{k-1}=\sum_{i=1}^{\infty}\frac{n_1n_2\cdots n_{k-1}q}{n_i}=\sum_{i=1}^{k-1}\frac{n_1\cdots n_{k-1}q}{n_i}+\sum_{i=k}^{\infty}\frac{n_1\cdots n_{k-1}q}{n_i}.$$

Now, $pn_1\cdots n_{k-1}$ and the finite sum on the right are integers, but

$$\sum_{i=k}^{\infty}\frac{n_1n_2\cdots n_{k-1}q}{n_i}$$

is, because of our choice of k, term for term less than the series $1/3+1/9+\cdots+1/3^i+\cdots$ whose sum is $\frac{1}{2}$. By this contradiction the proposed result is established.

Also solved by Joshua Barlaz, N. J. Fine, W. Fulks, Stanley Katz and A. M. Peiser, J. B. Kelly, R. S. Lehman, Norman Miller, Leo Moser, Alex Rosenberg, and Ernest Trost.

Binomial Coefficients and Greatest Integer Function

4322 [1948, 642]. *Proposed by Pedro A. Piza, San Juan, Puerto Rico.*

If p is a prime, prove that

$$\binom{n}{p}\equiv[n/p]\pmod{p},$$

where $\binom{n}{p}$ and $[n/p]$ have their usual meanings.

Prove also that when $[n/p]$ contains p^s as a factor, then $\binom{n}{p}$ is divisible by p^s.

Solution by Joshua Barlaz, Rutgers University. Consider the n consecutive numbers, $n, n-1, \cdots, n-p+1$. These are congruent in some order to the numbers $1, 2, \cdots, p \pmod{p}$. Moreover, one of them, say N, is divisible by p, whence $[n/p]=N/p$. The remaining set, when we remove N, will be congruent to $1, 2, \cdots, p-1 \pmod{p}$ in some order.

Let Π equal the product of this set, that is

$$\Pi = \frac{n(n-1)\cdots(n-p+1)}{N}.$$

Then

$$\Pi \equiv (p-1)! \pmod{p}.$$

Multiplying by N/p, we have

$$\frac{N\Pi}{p} \equiv \frac{(p-1)!N}{p} \pmod{p}, \tag{1}$$

and dividing by $(p-1)!$ which is relatively prime to p, we have

$$\frac{N\Pi}{p!} \equiv \frac{N}{p} \pmod{p}, \tag{2}$$

where $N\Pi/p!$ is the integer $\binom{n}{p}$. This is the Proposer's first result.

If $N/p=[n/p]$ is divisible by p^s, then the congruences (1) and (2) become true modulo p^{s+1}. That $N\Pi/p=\binom{n}{p}$ is also divisible by p^s is a corollary.

Also solved by R. Bott, D. H. Browne, I. C. Diehm, N. J. Fine, Seymour Haber, O. H. Hoke, R. S. Lehman, L. C. Mathewson, Leo Moser, Alex Rosenberg, Azriel Rosenfeld, N. C. Scholomiti, Robert Steinberg, and Ernest Trost.

Editorial Note. Fine, in his solution, notes that the present results are corollaries of the following theorem of Lucas: *If p is a prime, and*

$$n = n_0 + n_1 p + \cdots + n_k p^k \qquad (0 \leqq n_r < p),$$

$$m = m_0 + m_1 p + \cdots + m_k p^k \qquad (0 \leqq m_r < p),$$

then

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_k}{m_k} \pmod{p}.$$

For proof see Fine, *Binomial Coefficients Modulo a Prime*, this MONTHLY, Dec. 1947.

A Double Limit

4324 [1948, 642]. *Proposed by W. O. Pennell, Exeter, New Hampshire.*

When $x \geqq 0$ what is the limit as $n \to \infty$ of

$$\phi(n, x) = e^{-x^n}\left[1 + \frac{x^n}{\frac{1}{n}+1} + \frac{x^{2n}}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)} + \frac{x^{3n}}{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)} + \cdots\right]?$$

Solution by Stanley Katz and A. M. Peiser, New York City. It is clear that we may take $x > 0$. Let

$$g_n(x) = xe^{x^n}\phi(n, x) = x + \sum_{k=1}^{\infty} \frac{n^k x^{kn+1}}{(n+1)\cdots(kn+1)}.$$

Then

$$\begin{aligned} g_n'(x) &= 1 + nx^n + \sum_{k=2}^{\infty} \frac{n^k x^{kn}}{(n+1)\cdots((k-1)n+1)} \\ &= 1 + nx^{n-1}g_n(x). \end{aligned}$$

Since $g_n(0) = 0$, the solution of this differential equation is

$$g_n(x)e^{-x^n} = \int_0^x e^{-u^n}du,$$

$$\phi(n, x) = \frac{1}{x}\int_0^x e^{-u^n}du.$$

For $0 < x \leqq 1$,

$$\lim_{n\to\infty} \phi(n, x) = \frac{1}{x}\int_0^x \lim_{n\to\infty} e^{-u^n}du = 1.$$

For $x \geqq 1$,

$$\lim_{n\to\infty} \phi(n, x) = \frac{1}{x}\int_0^1 \lim_{n\to\infty} e^{-u^n}du + \frac{1}{x}\int_1^x \lim_{n\to\infty} e^{-u^n}du = \frac{1}{x}.$$

Also solved by Robert Steinberg, N. J. Fine, and the Proposer.

Powers of 2

4327 [1949, 39]. *Proposed by R. J. Walker, Cornell University*

In attempting to solve the preceding problem* we tried to show that for some positive integer s no power of 2 had its last s digits all powers of 2. Show that this attempt had to fail; in particular, show that for each s there exist powers of 2 each of whose last s digits is either 1 or 2.

Solution by E. P. Starke, Rutgers University. The proof proceeds by induction. Noting that $2^5=32$ and $2^9=512$, let us suppose that 2^n is a number whose last s digits are exclusively 1's and 2's, so that

$$2^n \equiv z \pmod{10^s},$$

where the s-digit number z is composed entirely of 1's and 2's. From the properties of the Euler ϕ-function we have

$$2^{\alpha(s)} \equiv 1 \pmod{5^s},$$

where $\alpha(s)=\phi(5^s)$, and hence

$$2^{\alpha(s)}A \equiv A \pmod{2\cdot 10^s} \tag{1}$$

provided A is divisible by 2^{s+1}. Thus A and $2^{\alpha(s)}A$ have the same s final digits, and their critical digits are of like parity. (We shall refer to the digit preceding the s final digits of any number as its critical digit.)

Now consider the numbers

$$2^{n+k\alpha(s)}, \qquad k = 0, 1, 2, 3, 4. \tag{2}$$

These five numbers have distinct critical digits; for, otherwise we would have

$$2^{n+k\alpha(s)} \equiv 2^{n+k'\alpha(s)} \pmod{10^{s+1}}, \qquad 5 > k > k' \geqq 0,$$

whence

$$2^{(k-k')\alpha(s)} \equiv 1 \pmod{5^{s+1}};$$

but this is impossible since $(k-k')\alpha(s) < 5\phi(5^s) = \phi(5^{s+1})$, whereas the fact that 2 is a primitive root, mod 5, implies

$$2^r \not\equiv 1 \pmod{5^{s+1}} \qquad \text{for } r < \phi(5^{s+1}).$$

This means that if the critical digit of 2^n is odd (or even) then the critical digits of the numbers (2) are, in some order, 1, 3, 5, 7, 9, (or 0, 2, 4, 6, 8). Since by (1) the five numbers (2) have the same s final digits as 2^n, one of them has its $s+1$ final digits exclusively 1's and 2's. This completes the induction.

* No. 4326 by the same proposer: Prove or disprove: 128 is the only power of 2 of two or more digits each of which is a power of 2. (No solution or discussion has been submitted.)

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

The Mathematical Analysis of Logic. By George Boole, Macmillan, Barclay, and Macmillan, 1847. Reprinted. Philosophical Library, New York, 1948. 6+82 pp. $3.75.

A hundred years ago logic was still a "literary" science, about which people could talk at length, and oftentimes foolishly, without its being possible for anyone to prove them wrong. Like ethics and politics, logic rested on no solid foundation, and almost everything was subject for controversy.

But just when political thought was taking a turn for the worse (with the publication of the *Communist Manifesto* in 1848), the prospects for logic were beginning to brighten. Boole and De Morgan were loosening the bonds of scholastic philosophy—and showing that logic could be made into an exact mathematical science.

The historically minded reader can get a good notion of the difficulties confronting these early logicians, from Boole's *Mathematical Analysis of Logic*, now happily reprinted by the Philosophical Library. How difficult it must have been to devise an appropriate symbolism for logic when one had first to present arguments in favor of using any symbolism at all, even in mathematics—arguments against the objections which "have lately been very strongly urged against this practice, on the ground, that by obviating the necessity of thought, and substituting a reference to general formulae in the room of personal effort, it tends to weaken the reasoning faculties."!

The less serious reader will find a source of amusement in some of Boole's queer proofs, and a ground of self-satisfaction in living in so superior an age. For such a purpose, I cannot recommend anything better than the weird proof (page 60) of the law of expansion for Boolean functions: a proof which proceeds by expanding an arbitrary Boolean function in a Maclaurin's series and this, when the author has not defined derivatives of Boolean functions nor division of them by integers.

J. C. C. MCKINSEY

Principles of Mechanics. By J. L. Synge and B. A. Griffith. Second Edition. New York, McGraw-Hill, 1949. 16+530 pages. $5.00.

That a second edition of this intermediate text in theoretical mechanics should be required after a relatively short interval is evidence of its deserved popularity. The authors state in the preface that a textbook on mechanics must attempt to reconcile "two opposing goals—the reduction of the subject to a compact and classified form and its exposition with sufficient fullness to make the arguments easy to follow." Indeed, any textbook upon this most elegant

and complete branch of mathematical physics must also embody a compromise among three points of view: that of the mathematician, who cultivates the general theory of mechanics as a deductive system, and directs his efforts mainly toward transformation theory and questions of existence and stability; that of the physicist, who, seeking to describe the behavior of actual matter, desires on the one hand a thorough familiarity with the simplest situations in mass-point mechanics to serve as a starting point for physical models of crystals, atoms, *etc.*, and on the other, a general formulation of classical mechanics which is suggestive for the newer theories, so as to leave it behind at the earliest opportunity in his pursuit of high velocities and small dimensions; and that of the engineer, who desires to turn his equations to the design of practical devices. The resulting compromise leads to the science of *rational mechanics*, which many European universities deservedly recognize by separate chairs—a science unfortunately neglected on this continent. Every good book on mechanics represents this compromise and the present one is no exception. The following discussion is to be seen in the light of these remarks, not being meant as criticism of the general plan, but rather as presentation of an alternative point of view with regard to a few details.

In reviewing the first edition, J. W. Campbell (*Bulletin of The Am. Math. Soc.*, 49, 208–211, (1943)) offered five principal criticisms: (1) the basic concepts are insufficiently discussed for a first course; (2) vectors are defined incorrectly (or at best inconsistently) (pp. 18, 20); (3) the term "impulsive force" is used instead of "impulse," although this quantity is not a kind of force at all and is not even of the dimensions of force; (4) it is incorrectly stated (p. 88) that the coefficient of static friction $\mu \leqq 1$ experimentally and (5) while the theory of dimensions is treated in an appendix, there is little use of dimensional principles in the text. The authors have chosen to accept this criticism only in the fifth instance, where it was least well taken, and thus have inserted a number of remarks and easy exercises concerning dimensions at various points. In fact, in the theoretical mechanics of mass-points and rigid bodies the theory of dimensions is less important than in any other branch of mathematical physics, and may be altogether omitted without any loss. Once having chosen to mention it however, it is unfortunate that the authors merely reproduce the usual vague remarks to be found in physics and engineering texts, not going even so far as the pi-theorem, instead of showing that indeed dimensional theory is a simple clearly defined algebra like the algebra of real numbers, which we all use in a more or less unconscious fashion—an algebra underlying all mathematical physics very much as the algebra of logic underlies mathematics. As for the first criticism, perhaps the authors have noticed that it is only quite advanced students who begin to understand the fundamentals of a subject, whatever may or may not be said in an elementary course, whose main function is to give the student facility and structural orderliness in handling the facts with which one builds a science "comme un maison avec des pierres," without which he cannot appreciate the strength or weakness of those facts. It is unfortunate, however, that the authors did not see fit to be more careful regarding the purely mathe-

matical definitions, as in the case of a vector. It would seem also that here the student might first learn that every theory is an abstract mathematical model, rather than being allowed to fancy that he is somehow dealing directly with the phenomena the theory is designed to describe.

In his review of the first edition W. Prager [*Math. Rev.* 3, 213 (1942)] remarked upon the sparing use of vectorial methods, and in particular upon the fact that the authors' eschewance of tensorial or dyadic notations renders their treatment of the motion of rigid bodies quite lengthy. In the first edition the authors were content to state that since the equation of any ellipsoid could be reduced to standard form by a proper choice of axes, the existence of principal axes of inertia followed. In the second edition they have given a proof of this statement, but it is unfortunate that in what may be the student's first encounter with the characteristic value problem he will have to read a long and special treatment of a special case, instead of learning once and for all a general theorem that would be of frequent usefulness. In any case, the moments and products of inertia are brought in by the heels rather than shown to arise most naturally in the problem of reducing the equations of rigid dynamics to a simple and perspicuous form.

The second edition differs from the first mainly in the addition of a few words of explanation (as in the case of d'Alembert's principle), a few figures, and a few problems, amounting to fifteen pages in all. In addition to the one major change already mentioned, the only other is the insertion of a careful and very interesting treatment (pp. 388–403) of the motion of a charged particle near the axis of symmetry of an electromagnetic field, including applications to the electromagnetic lens.

To return again to the attitude expressed in the opening sentences of this review, there is one respect in which the book leans too far toward the engineering side. Lagrange's equations are introduced only after 450 pages, the applications made of them are not important, and their extension to rigid dynamics is not mentioned. The authors say "However, it seems probable that as time passes the method of Lagrange will work its way from the end to the beginning of textbooks on mechanics. The easier, but more cumbrous, methods are becoming a luxury for which we cannot afford the time." It is unfortunate that the authors did not actualize their own position. Since many a physics curriculum nowadays affords only one course in mechanics, it would seem that Hamilton's canonical equations and the Hamilton-Jacobi equation should be included, although no mention of them occurs in this book.

An elementary textbook cannot be loaded with references and attributions, but it would seem a pleasant duty of the writers of such a book to connect a number of the methods and theorems with the great names responsible for them. Although there is hardly a great mathematician or physicist from 1650 to 1850 who did not make some important contribution to their subject, the authors mention only about twenty-five classical names, and those very sparingly.

Any textbook serves principally as a guide for the instructor and students,

who may make a reasonable number of additions or omissions. In this respect the present book is admirable, for the additions one is likely to suggest are few, and the omissions only such as time enforces.

C. TRUESDELL

Introduction to Analytic Geometry and The Calculus. By H. M. Dadourian. New York, The Ronald Press, 1949. 10+246 pages. $3.25.

This is a textbook designed "for use in combined courses of Freshman mathematics, such as are offered by an increasing number of colleges and universities." For a terminal course, the author points out that trigonometry is not presupposed except in a single section (which can be omitted), thus making possible an exclusive emphasis on analytic geometry and calculus. Otherwise, for the student who will study mathematics further, a brief course in trigonometry would seem to be desirable before undertaking the subject matter of this text.

An instructor in choosing a textbook will, naturally, take into account not only the subject matter he plans to teach but also the preparation and background of his students, that is, the general intellectual level of his audience. It should be noted, therefore, that the volume under review "is only an introduction to the simplest elements of analytic geometry and the calculus." And that the author has attempted to present these elements with a continuing and conscientious regard for the difficulties that students, beginning in these fields, always meet.

Following an introductory chapter on how to study mathematics and on solving problems, the section on analytic geometry includes chapters on functions and graphs, the linear function and the straight line, and the quadratic function and conics. The author's restriction to the "simplest elements" has resulted apparently in the omission of any study of the general second degree equation, and other omissions of material usually considered in a first course will be noted. Nevertheless, there is considerable information for the student here. This reviewer, however, would have preferred that some of this information might have been the result of "derivation" rather than mere "statement of fact," and, as a consequence, the author's admonition for the student "to think" the better motivated.

Five chapters on the differential calculus cover fairly well the usual theory and applications of a first course, at least as well as one can expect when limited to the derivative of powers of algebraic functions. Under an analogous limitation, the integral calculus is developed in theory and applications in the four final chapters.

Keeping in mind the author's aim to present this introductory material to college freshmen, which is of course an entirely reasonable and justifiable aim, it would nonetheless seem pertinent here to criticize the execution on several counts: The lack of *explicit* definitions in many cases where the consequences are based entirely on the definition (cf., pp. 165, 182); loose or incomplete state-

ments and definitions (cf., pp. 77, 81); and a tendency (mentioned above) to state facts rather than to deduce results. In particular, the last chapter would seem to suffer especially in these respects (cf., "negative mass," p. 199, and Sections 142–143).

At the same time one can have only praise for the author's choice of illustrative material, his clear exposition of this material, and his problem lists. In general, the text reads well and possesses an admirable conciseness of statement. Only one misprint was observed (p. 110), but one which will probably not be condoned by the devotees of Lewis Carroll!

B. C. PATTERSON

NEW BOOKS RECEIVED

College Algebra. By. E. B. Miller and R. M. Thrall. New York, The Ronald Press. 493 pp. $3.75.

Plane and Spherical Trigonometry. By M. Richardson. New York, Macmillan, 1950. 15+138 pp. $3.75 ($3.40, without tables).

Modern College Geometry. By D. R. Davis. Cambridge, Addison-Wesley, 1949. 10+238 pp. $4.50.

Elements of Mathematical Analysis. By S. E. Urner and W. B. Orange. Boston, Ginn and Company, 1950. xi+562 pp. $4.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1948–1949

Pi Mu Epsilon, University of Georgia

The following papers were presented before the *Pi Mu Epsilon* fraternity at the University of Georgia during 1948–49:

Honor courses at the University of Georgia, by Prof. Tomlinson Fort

Sturms' theorem, by Prof. Tomlinson Fort

Unit fractions of early mathematics, by Howard Burnette

The geometry of the triangle and some of its applications, by Prof. Hugh Henry of the Department of Physics

The history of Pi Mu Epsilon, by Prof. Tomlinson Fort

Certain theorems on matrices, by Prof. J. A. Ward

The symbiosis of knowledge, by Prof. E. H. Dixon of the Department of Physics

G. H. Hardy's solution of a problem in genetics, by Prof. R. J. Levit

Geometry, a series of colloquium lectures, by Prof. C. C. MacDuffee of the University of Wisconsin.

A colloquium lecture was also given each month by one of the professors.

The officers for the coming year are: Director, George Peters; Vice-Director, William Shy; Secretary-Treasurer, Ann Taylor; Faculty Advisor, Prof. Alonzo Cohen; Corresponding Secretary, Prof. W. S. Beckwith.

Mathematics Club, University of Buffalo

The topics discussed by the University of Buffalo *Mathematics Club* were:

Construction by compass alone, by C. Collins and M. Kallett

Geometric approach to some algebra expressions, by C. Christenson

Mathematical curiosities, by George Rudroff

A collinear set in a lattice of points, by George Rudroff

Diophantus solves an algebra problem, by Thomas Huestis

Construction of a differential analyzer, by Brent Haslam

Determination of elementary symmetrical polynomials, by Brent Haslam

Arithmetic short cuts, by Richard Smurthwaite and Harold Cutcher

Derivation of formulas for obtaining the sums of powers of the roots, and the elementary symmetrical polynomials in terms of each other, by Albert Fadell.

The officers for the year 1949–50 are: President, Roger Hill; Vice-President, Richard Smurthwaite; Treasurer, Fred Gregory; Secretary, Wilbur Hastings; Sponsor, Miss June McArtney.

Kappa Mu Epsilon, Drake University

The activities of *Iowa Beta* chapter of *Kappa Mu Epsilon* for the past year were devoted to reorganization meetings, initiations, business meetings and a banquet. A modification of the chapter's by-laws took several meetings.

Three delegates attended the national convention of Kappa Mu Epsilon at Topeka, Kansas.

The officers elected are: President, William Lafferty; Secretary, June Kagawa; Treasurer, Marvin Van Cleave; Corresponding Secretary, Prof. Earle Canfield.

Pi Mu Epsilon, University of Wisconsin

The *Wisconsin Beta* chapter of *Pi Mu Epsilon*, inactive during the war period, was reorganized during the academic year 1947–48 as a graduate student club to which a number of graduating seniors are elected each year in honor of their high scholarship.

The program of the chapter during 1948–49 included the following talks:

Computation by punched cards, by Fred Gruenberger

Atomic elements in Boolean algebras, by Joseph Engel

Duality in differential calculus and projective geometry, by Rodney Hood

Loops and their isotopes, by Donald Norton

The P-adic and G-adic number systems, by Richard Andree.

A get-acquainted open house was held for the mathematics faculty, graduate students and guests. Other social events included an initiation banquet and another party.

Officers for 1948–49 were: President, James Renno; Vice-President, Herman Cohen; Secretary-Treasurer, Violet Hachmeister; Faculty Adviser, Prof. R. E. Fullerton.

Mathematics Society, University of Miami

The *Mathematics Society* of the University of Miami was organized in July 1948. During the year 1948–49 fifteen regular meetings and a banquet were held. Among the papers presented were:

Rhind papyrus, by W. G. Franzen

Probability curve, by L. G. Barbee

Boolean algebra, by Prof. Ira Rosenbaum

Natural law of growth and the number e, by D. J. R. Foulis

Magic squares, by Miss Ethelmae Miller

Notation in differential and integral calculus, by J. H. Maecher.

The officers for 1948–49 were: President, William Franzen; Vice-President, Eugene P. Goldberg; Recording Secretary, Marjorie Stern; Corresponding Secretary, Gloria Lichtenstein; Treasurers, Arlo Shaver and John Maecher; Faculty Advisor, Prof. Mary Dean Clement.

Kappa Mu Epsilon, Nebraska State Teachers College

The *Nebraska Alpha* chapter of *Kappa Mu Epsilon* held two initiation meetings during 1948–49. Other meetings consisted of business meetings and the following programs:

Salesmanship, by Donald Simpson

Certain aspects of mathematics in high school, by President V. P. Morey.

Seven members of *Kappa Mu Epsilon* attended the State meeting of the National Council of Teachers of Mathematics at which Dr. Morey was the luncheon speaker.

The only new officer elected for the coming year is James Callaway, who is the Organizing Chairman.

Mathematics Club, Berea College

The *Mathematics Club* of *Berea College* had regular monthly meetings. Some of the topics discussed were:

Possibilities for assistantships in graduate school, by Donald W. Pugsley

The abacus, by George S. Noss

Oddities in mathematics, by Jack Hale

Relationship between philosophy and mathematics, by W. G. Ross

Short-cuts in mathematics, by V. D. Roberts.

The elected officers were: President, Howard Stevens; Vice-President, Robert Manning; Secretary-Treasurer, Jacky Gene Hopper.

Pi Mu Epsilon, University of Nebraska

The *Nebraska Alpha* chapter of *Pi Mu Epsilon* reports that, for the year 1948–49, eight meetings were held including business meetings and the two initiation banquets. The following talks were given:

Special theory of relativity, by Prof. Theodore Jorgensen
Boolean algebra, by Eugene Luchei
Quantum mechanics, by Dr. J. H. Cohn
Calculating machines, by Fred Pelton
Television, by Mr. D. D. Holmes
"*Murder incorporated*," by Dr. Frederick Ludwig.

The chapter sponsored classes to extend aid in mathematics to the slower students. Members of the chapter volunteered their services as instructors. The plan proved highly satisfactory as it benefited both the students and the tutors.

A pamphlet was prepared which introduced *Pi Mu Epsilon* to students not familiar with the organization.

The annual freshman and sophomore competitive mathematics examinations were held; the sophomore award was won by Robert Kleppinger, who also won the freshman award last year, and Chris Kuyatt was the winner of the freshman award.

The newly elected officers to serve for the coming year are: President, Cecil Doubt; Vice-President, Robert Ceasari; Secretary, Donald Cochran; Treasurer, David Olive; Faculty Advisor, Dr. W. G. Leavitt.

Kappa Mu Epsilon, Baldwin-Wallace College

Papers presented to the *Ohio Gamma* chapter of *Kappa Mu Epsilon* during 1948–49 were:

Boolean algebra, by Donald Parrish
The mathematics of audio amplification, by Ralph Paul
The duodecimal system, by Robert Jungles
Report of the national convention of Kappa Mu Epsilon, by Donald Parrish and Howard Mielke
Probability—the law of averages, by Dr. R. W. Wagner of Oberlin College
Mathematics and aviation, by Cook Cleland.

The annual banquet of *Kappa Mu Epsilon* was held in May at which Dr. R. W. Wagner spoke.

The society visited the National Advisory Committee on Aeronautics Laboratory at the Cleveland Airport. Various applications for mathematics were shown. One of the many interesting things seen was the world's largest supersonic wind tunnel.

Officers elected for 1949–50 are: President, Fred Rakowsky; Vice-President, Dwain Diller; Secretary, Anita Constantine; Treasurer, Curt Liebert; Corresponding Secretary and Sponsor, Prof. Paul Annear.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

CONFERENCES AND SYMPOSIA

A Symposium on the Theory of Electromagnetic Wave Propagation will be held at New York University on June 6–8, 1950.

The Second Berkeley Symposium on Statistics and Probability will be held at the University of California, Berkeley, from July 31 to August 12. This meeting will be given to theory and to several fields of application. Up to the present time the following persons have accepted invitations to participate in the Symposium: Professors W. G. Cochran, Harald Cramér, J. L. Dobb, G. C. Evans, William Feller, Harold Hotelling, Mark Kac, Paul Lévy and Abraham Wald.

The University of Chicago announces that a Conference on Topology will be held on May 24–27, 1950.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1950:

Boston University. May 31 to July 8: Dr. Giever, topics in the theory of numbers; Dr. Scheid, calculus of variations. July 10 to August 19: Professor Johanson, introduction to higher algebra; Professor Sobczyk, introduction to modern mathematics, seminar in analysis; Professor Chand, the design of experiments, introduction to mathematical statistics.

Catholic University of America. June 26 to August 5: Professor Rice, advanced calculus and statistics; Professor Finan, higher algebra; Mr. Clark, partial differential equations of mathematical physics; Dr. Markowitz, mathematical astronomy I and II; Dr. Moller, theory of equations; Professor Ramler, college geometry, differential equations and synthetic projective geometry.

Columbia University. July 10 to August 18: Professor Ritt, differential equations; Professor Levi, theory of functions of a complex variable; Professor Koopman, Fourier series and integrals; Professor Eilenberg, introduction to modern topology; Professor Dieudonné, introduction to classical groups; Professor Lorch, introduction to mathematical logic.

Northwestern University. The following courses will be offered in both the six-week session, June 23 to August 5, and the nine-week session, June 23 to August 26: differential equations, fundamental concepts of analysis; definite integrals; non-euclidean geometry; theory of statistics; introduction to the theory of numbers; algebra for teachers; the history and teaching of mathematics; functions of a complex variable; the continuum; introduction to modern algebra; seminar; independent study; thesis. A course, the teaching of mathematics in the secondary school, will be offered during the six-week session only.

University of Arkansas. June 5 to July 14: Professor Richardson, college geometry, theory of equations; Professor Perry, advanced calculus, Laplace transform; Mr. Hull, differential equations. July 17 to August 26: Professor Gundlach, history of mathematics, vector analysis.

University of California, Berkeley. Department of Mathematics. June 19 to July 29: Professor Besicovitch, extremal problems in geometry; Professor McDonald, coordination in mathematics for secondary schools.

University of California, Berkeley. Department of Mathematical Statistics. June 19 to July 29: Professor Lehmann, recent developments in the theory of estimation; Professor Cochran, design of experiments, seminar on current problems in analysis of variance; Professor Lévy, random variables and random functions, seminar in probability theory.

University of Chicago. June 26 to September 2: Professor Barnard, analysis E, vector analysis; Professor Kaplansky, fundamental concepts of geometry, algebra II; Professor Schilling, algebra V, theory of rings; Professor Halmos, theory of sets, seminar in probability; Professor Zygmund, Lebesque and Stieltjes integrals, regular functions in the unit circle; Professor Segal, partial differential equations; Professor Stone, approximation in the complex domain; Professor Weil, functions of several complex variables, historical seminar.

University of Oklahoma. June 8 to August 7: Professor Court, college geometry, history of mathematics; Dr. Huff, ordinary differential equations; Dr. Andree, solid analytical geometry; Professor Bernhart, ordinary and partial differential equations; Mr. LaFon, advanced calculus; Professor Grau, theory of groups; Professor Springer, partial differential equations; Professor Goffman, infinite series; Professor Whitney, celestial mechanics.

University of Southern California. June 26 to August 4: Professor Albert, modern higher algebra, seminar in algebra; Professor Dugundji, advanced calculus; Professor Henkin, introduction to mathematical logic; Professor Phillips, integral equations; Dr. Shniad, vector analysis; Professor White, advanced calculus; Professor Whiteman, theory of numbers, matrix theory.

University of Virginia. June 26 to August 19: Professor Klee, advanced calculus and applied mathematics; Professor Botts, differential equations and applied mathematics; Professor Whyburn, analytic topology; Professor Floyd, advanced differential equations.

PERSONAL ITEMS

Professor C. O. Oakley of Haverford College and Professor G. C. Webber of the University of Delaware were appointed to serve as delegates of the Association at the Annual Meeting of the American Academy of Political and Social Science which was held in Philadelphia on April 14–15, 1950.

Professor W. G. Scobert of the University of Idaho represented the Association at the Centennial Celebration of the University of Utah, February 27–28, 1950.

Professor Einar Hille of Yale University has been elected Vice-President of the American Association for the Advancement of Science and Chairman of the Mathematics Section; Professor R. W. Brink, University of Minnesota, has been re-elected Secretary of the Mathematics Section.

Southern Illinois University announces the following appointments for the 1950 summer session: Professor Emeritus W. B. Carver of Cornell University as Professor of Mathematics; Associate Professor T. H. Southard of Wayne University as Associate Professor of Mathematics.

Stanford University reports that the following persons will be associated with the Mathematics Department during the Summer Quarter, 1950: Harold Davenport, University College, London; Werner Fenchel, Polytechnical School, Copenhagen; Paul Garabedian, University of California, Berkeley; W. K. Hayman, Exeter College, England; Charles Loewner, Syracuse University; W. W. Rogosinski, King's College, Newcastle, England; H. L. Royden, Harvard University.

University of California, Berkeley, announces the following promotions, appointments and leaves of absence for the year 1949–50: Professor C. B. Morrey has been appointed Chairman of the Department of Mathematics; Associate Professors Sophia L. McDonald, A. P. Morse, R. M. Robinson have been promoted to professorships; Dr. S. P. Diliberto, Dr. J. L. Hodges, Jr., and Dr. R. C. James have been promoted to assistant professorships; Dr. H. B. Ribeiro, Marianne F. Smith and Virginia W. Wakerling have been promoted to instructorships; Dr. P. R. Garabedian has been appointed to an assistant professorship; R. M. Lakness, Ting-Kwan Pan and Esther Seiden have been appointed Lecturers; Dr. H. B. Mann has been appointed Visiting Professor; Professor D. H. Lehmer and Professor R. M. Robinson are on sabbatical leave for the year; Professor Jerzy Neyman is on leave for the spring term 1950 and will deliver lectures at University College, London, Institut Poincaré and University of Paris, Paris; Assistant Professor Charles Stein holds a National Research Council Fellowship and is spending his leave in Paris; Professor Alfred Tarski was on leave in March 1950 to lecture at University College, London, and the University of Amsterdam.

University of Florida reports: Assistant Professors B. F. Dostal and G. B. Lang have been promoted to associate professorships; Instructor R. G. Blake has been promoted to an assistant professorship.

University of Georgia announces the following: Dr. A. S. Householder of the Oak Ridge National Laboratory gave a series of talks on numerical methods to the mathematics colloquium on February 15–17, 1950; during March 27 to April 4 Professor J. L. Walsh of Harvard University gave a series of colloquium lectures on approximation by polynomials in the complex domain, a Sigma Xi lecture on surface navigation, and a Pi Mu Epsilon lecture on maximum and minimum problems.

Assistant Professor L. A. Aroian of Hunter College is on leave the spring semester. He is serving as Research Physicist in the Department of Electronics

and Guided Missiles of the Hughes Aircraft Company, Culver City, California, in charge of computations.

Professor R. V. Blair of Vanderbilt University was reported erroneously in this department, February, 1950, to have the rank of associate professor. He was promoted to a professorship in 1948.

Associate Professor N. R. Bryan of the University of Georgia has been appointed to an associate professorship at Clemson College.

Mr. Roland Deaux has assumed the editorship of *Mathesis*.

Instructor Watson Fulks of the University of Minnesota has been promoted to an assistant professorship.

Mr. A. G. Hansen has been appointed to an instructorship at the University of Maryland.

Professor Eberhard Hopf, formerly of New York University, has been appointed to a professorship at Indiana University.

Mr. D. W. Hullinghorst has accepted a position as director of the Civil Works Investigation Project, South Pacific Division, Engineering Corps, Oakland Army Base, Oakland, California.

Mr. Sydney Kaplan, formerly a research mathematician in the Applied Mathematics Division, Naval Ordnance Laboratory, has accepted the position of Statistician, Bureau of Employment Security, United States Department of Labor.

Dr. Knox Millsaps of the Air Materiel Command, Wright Field, has been appointed to a professorship of theoretical physics at the Alabama Polytechnic Institute.

Mr. S. F. Neustadter has been appointed to an instructorship at Harvard University.

Professor Rufus Oldenburger of DePaul University has resigned to devote more time to the Woodward Governor Company, Rockford, Illinois, as mathematician-engineer.

Dr. Irene Price, formerly statistician for the United States Air Force, has been transferred to Holloman Air Force Base, New Mexico, to direct mathematical research on guided missiles.

Dr. L. T. Ratner has been appointed to an assistant professorship at Vanderbilt University.

Professor C. J. Rees, director of graduate study, University of Delaware, has been appointed Director of Summer Session.

Dr. H. M. Schwartz of Brookhaven National Laboratory has accepted an associate professorship at the University of Arkansas.

Dr. Edward Silverman of the Institute for Advanced Study has been appointed to a visiting assistant professorship at Kenyon College.

Assistant Professor F. E. Smith of Brooklyn College has been promoted to an associate professorship.

Professor A. W. Tucker of Princeton University has been spending the

academic year 1949–50 at Stanford University as Visiting Professor of Mathematics.

Professor W. D. A. Westfall of the University of Missouri has retired.

Miss Bertha E. Frankenbush, emeritus high school teacher of New Orleans, Louisiana, died February 7, 1950.

Professor Emeritus W. J. Humphreys of George Washington University died November 10, 1949.

Professor Emeritus F. M. McGaw of Cornell College died on February 6, 1950. He was a charter member of the Association.

Mr. A. Mineur, editor of *Mathesis*, died on February 4, 1950.

Professor Emeritus C. E. Wilder, Dartmouth College, who was a charter member of the Association, died on February 9, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE OKLAHOMA SECTION

The October meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Friday, October 14, 1949. Professor R. W. Veatch, Chairman of the Section, presided.

One hundred twenty persons attended the meeting, including the following forty-three members of the Association: E. F. Allen, R. V. Andree, Arthur Bernhart, J. C. Brixey, W. C. Brown, H. N. Carter, R. L. Caskey, N. A. Court, R. B. Deal, A. H. Diamond, R. C. Dragoo, J. R. Ellis, D. A. Gorsline, L. D. Gregory, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. E. Heimann, L. R. Holland, S. L. Hull, P. W. M. John, W. G. Johnson, J. T. Krattiger, W. D. Krentel, J. E. LaFon, M. L. Lawson, Eunice Lewis, H. W. Linscheid, R. D. McDole, Dora McFarland, J. C. C. McKinsey, R. D. McKnelly, G. E. Meador, R. R. Murphy, C. L. Perry, C. J. Pipes, C. M. Pirrong, Herbert Scholz, H. W. Smith, T. C. Smith, C. E. Springer, R. W. Veatch, J. H. Zant.

At the business session the following officers were elected: Chairman, J. T. Krattiger, Southeastern State College; Vice-Chairman, S. L. Hull, University of Arkansas; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following seven papers:

1. *Orientation of the new teacher*, by L. W. Lavengood, Director of Mathematics, Tulsa Public Schools, introduced by Professor Veatch.

Mr. Lavengood discussed methods used to help orient new teachers in the Tulsa Public Schools

and pointed out that new teachers should be made to feel welcome in the community, and have a responsible part in the school organization.

2. *Essential and inessential transformations of point sets into n-spheres*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

Professor Hamilton gave a new proof of the theorem that every continuous transformation of an n-cell into an n-sphere is inessential, and discussed the relation between the concept of an inessential transformation and fixed point theorems.

3. *Old and new on the centroid*, by Professor N. A. Court, University of Oklahoma.

A historical discussion of the notion of centroid, and a new treatment of centroid, were given in this paper.

4. *Axiomatic foundations of the theory of games*, by Professor J. C. C. McKinsey, Oklahoma A. and M. College.

5. *L'Hospital's rule*, by Professor Arthur Bernhart, University of Oklahoma.

Criticisms were given of the treatments of L'Hospital's rule appearing in several of the popular elementary calculus texts.

6. *The summation of a Dirichlet series*, by Professor H. N. Carter, University of Tulsa.

The ordinary and general Dirichlet series were defined, and a statement was made regarding their introduction into analysis and application in analytic number theory.

7. *Contributions of Hilbert space theory to applied mathematics*, by Professor A. H. Diamond, Oklahoma A. and M. College.

Hilbert space theory was applied to the problem of determining the frequencies of vibration of a square clamped plate. The procedure consisted of a generalization by Aronszajn of a method devised by A. Weinstein for finding lower bounds for the eigenvalues.

J. C. BRIXEY, *Secretary*

OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at the University of North Dakota in Grand Forks, North Dakota, on Saturday, October 15, 1949. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors R. C. Staley, A. G. Hill and Kenneth May (Chairman of the Section) presided at the respective sessions.

Sixty-one persons attended the meeting, including the following twenty-five members of the Association: H. M. Anderson, F. J. Arena, L. E. Bush, E. J. Camp, Sister Seraphim Gibbons, Ruby M. Grimes, K. L. Hankerson, A. G. Hill, Karlis Kaufmanis, S. L. Mason, Kenneth May, W. H. McBride, W. R. McEwen, A. G. Montgomery, S. Mundhjeld, J. C. Peterson, P. A. Rognlie, L. W. Sheridan, F. C. Smith, I. W. Smith, R. C. Staley, Florence S. Stennes, A. G. Swanson, E. L. Swanson, Christine Westgate.

Professor E. J. Camp made a progress report for the Committee investigating

the advisability of establishing a high school mathematics contest. At the request of this Committee, a resolution was passed committing the Section to the sponsorship of such a high school contest as soon as arrangements can be made for it. Another resolution was passed authorizing the Executive Committee of the Section to act for the Section on all questions arising between meetings of the Section in connection with the arrangement of such a contest.

By invitation of the Executive Committee, Professor Peter Scherk of the University of Saskatchewan delivered an address at the morning session. The title of his address was *What are Tensors?* An abstract of this address follows:

A complete vector space is a vector space over the real field and with a scalar product such that the null-vector is the only vector perpendicular to the whole space. In the first part of his talk, Professor Scherk showed that the vectors in euclidean 3-space form such a space. Making use of ideas of Bourbaki's "algébre multilinéaire," he then developed the tensor algebra in an arbitrary complete vector space. The tensors of order p, for example, may be introduced as finite sums of ordered p-tuples of vectors, provided the equality of two such sums is defined in a suitable fashion.

The following eight short papers were presented:

1. *The star of Bethlehem from the astronomical point of view*, by Professor Karlis Kaufmanis, Gustavus Adolphus College.

In this paper, the author discussed several attempts to explain the star mentioned in the second chapter of St. Matthew as an astronomical phenomenon. He emphasized particularly the amazing agreement between the message of the Gospel and the conjunction of the planets Jupiter and Saturn in the sign of the Fish, which took place in the year 7 B.C.

2. *Substandard endowments with graded amounts of insurance*, by Professor F. C. Smith, College of St. Thomas.

If n-year endowment contracts with standard premiums $P_{x\,\overline{n}|}$ and standard reserves ${}_tV_{x\,\overline{n}|}$ are issued to a group of applicants whose mortality will follow a $100k$ per cent mortality table, then it is well-known that the amount of the insurance benefit for the t-th year is

$$\frac{1 + (k-1)\,{}_tV_{x\,\overline{n}|}}{k}.$$

In this paper, the author presents the usual derivation of this result by use of Fackler's accumulation formula, and then gives a proof of the fact that the present value of the insurance benefits is equal to the present value of the premiums. The proof, which is believed to be new, depends upon showing that two rather complicated polynomials are identical.

3. *Further remarks on the estimation of a function of several variables by graphical techniques*, by Professor L. W. Sheridan, The College of St. Thomas.

If X is a function of several variables, its value can be estimated by combining the several variables into a single variable, Z, by graphic techniques. Next Z and X are grouped into classes, and a cumulative table, with X expressed in per cent, is formed. The set of curves obtained from the cumulative table gives an estimate of the most probable class in which X may occur, and additional information as to the probability of its occurrence in adjacent classes.

4. *Problem on crystal vibrations*, Professor R. H. Cook, South Dakota School of Mines (introduced by the Secretary).

The theoretical determination of frequency for normal modes of vibration of crystal plates with two parallel lateral boundaries is presented. The method is one in which the differential equations are exactly satisfied, and the boundary conditions are approximated. It is shown that the method gives results practically identical to those of infinite plate theory for very high harmonics.

5. *The adjusted premium method of determining surrender values*, by Mr. I. W. Smith, Actuary, Pioneer Mutual Life Insurance Company, Fargo, North Dakota.

Prior to January 1, 1948, the cost of surrender of a life insurance policy, regardless of type, was fixed at a maximum of twenty-five dollars per one thousand dollars of insurance, or two and one-half per cent of the policy. This restriction applied only to the first twenty years duration of the policy. The cost was a graduated scale decreasing from approximately twenty dollars the first policy year to zero the twentieth policy year. The surrender value was the full terminal reserve the twentieth policy year.

Competition was the basis for some life insurance companies to return the full terminal reserve the tenth policy year. Surrender practice became irregular, the only restriction being the maximum cost of two and one-half per cent of the amount of the policy.

With the adoption of the Commissioners Standard Ordinary Mortality Table, effective January 1, 1948, actuaries have derived a method of determining the surrender cost by the adjusted premium. The adjusted premium is a new term in actuarial science. It is a function of mortality, duration of policy, and a maximum cost. It is expressed algebraically for an ordinary life policy as,

$$A_{P_x} = \frac{A_x + 20 + .4A_{P_x} + .25A_{P_x}}{a_x}$$

with the limiting value of 40 per cent of A_{P_x} not greater than sixteen dollars, and 25 per cent of A_{P_x} for any type of policy not more than twenty-five per cent of the adjusted premium for an ordinary life policy nor greater than ten dollars.

When the adjusted premium has been determined, the cash value is calculated by the prospective method of valuation. There will be a surrender charge for the entire premium paying period for any policy.

6. *Some elementary determinations of Pythagorean numbers*, by Professor Ruby M. Grimes, North Dakota Agricultural College.

This speaker proved by means of trigonometric relations and also by algebra that any three integers in the ratios of $2ab$, a^2-b^2, a^2+b^2, $(a>b)$ can be the three sides of a right triangle. A demonstration of the same property was accomplished by use of analytic geometry, using the unit circle, and this generalizes the other two proofs.

7. *Complex planes and Pauli matrices*, by Professor W. L. Woodley, North Dakota Agricultural College (introduced by the Secretary).

This speaker began with the fact that the circle $x^2+y^2=a^2$ can be represented by the equation $u\bar{u}=a^2$, where $u=x+iy$, and $\bar{u}$ is the complex conjugate of u. This fact was made the basis of certain generalizations of which Pauli matrices are a special case. Pauli used these matrices in studying the spin of an election.

8. *On some properties of the trinomial coefficients*, by Professor F. J. Arena, North Dakota State College.

If n is a positive integer, and if

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n},$$

the following relation connects the trinomial coefficients a_Ω with the binomial coefficients ${}_nC_\Omega$:

$$a_\Omega = \sum_{i=b}^{\Omega} {}_iC_{2b-i} \cdot {}_nC_i, \qquad \text{if } \Omega = 2S$$

$$a_\Omega = \sum_{i=S+1}^{\Omega} {}_iC_{2S-1} \cdot {}_nC_i, \qquad \text{if } \Omega = 2S + 1.$$

Other properties are stated and proved.

L. E. BUSH, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Haverford College, Haverford, Pennsylvania, on Saturday, November 26, 1949. Professor G. E. Raynor, Chairman of the Section, presided.

There were sixty-eight present, including the following forty-eight members of the Association, C. B. Allendoerfer, L. J. Burton, P. A. Caris, John Cawley, J. O. Chellevold, E. J. Cogan, Mary Louise Constable, E. H. Cutler, James Elmer Davis, F. L. Dennis, Arnold Dresden, H. H. Goldstine, Theodore Hailperin, R. C. Haseltine, C. E. Heilmen, Hyman Kamel, J. C. Lanz, V. V. Latshaw, Joseph Lehner, Marguerite Lehr, F. L. Manning, Edith A. McDougle, S. S. McNeary, A. E. Meder, Jr., W. R. Murray, A. B. Neale, C. A. Nelson, C. O. Oakley, E. R. Ott, J. C. Oxtoby, G. E. Raynor, R. D. Schafer, I. J. Schoenberg, Francis Aloysius C. Sevier, Benjamin Slepin, J. C. Smith, W. M. Smith, R. R. Stoll, D. L. Thomsen, Jr., Michael Tikson, R. R. Townsend, R. M. Walter, G. C. Webber, Anna Pell Wheeler, Albert Wilansky, A. H. Wilson, R. H. Wilson, Jr., H. M. Zerbe.

At the business meeting the following officers were elected for the coming year: Chairman, G. C. Webber, University of Delaware; Secretary, C. O. Oakley, Haverford College. The Program Committee for the next meeting will be: Theodore Hailperin (Chairman), Lehigh University, Joseph Lehner, University of Pennsylvania, and C. B. Allendoerfer, Haverford College. The 1950 meeting of the Section will be held at Lehigh University on November 25.

The program consisted of the following papers:

1. *Some observations relative to mathematics in research and development organizations*, by Dr. A. D. Hestenes, Franklin Institute, introduced by the Secretary.

This paper dealt with the process by which equipment is developed and studies are made in typical industrial research and development organizations, and stressed the role a mathematician may perform in such processes. The speaker described various types of mathematics encountered in such organizations. An interesting locus problem arose in the course of the author's industrial research activities. This locus, whose equation is

$$r_1 - a = c\,(r_2 - b)$$

where a, b and c are constants, and r_1 and r_2 are distances from fixed points p_1 and p_2 respectively, is a generalization of the well known expressions for the circle, ellipse and hyperbola.

2. *Some problems in numerical analysis*, by Dr. H. H. Goldstine, Institute for Advanced Study.

There are four major sources of error in a numerical computation: those arising from the idealizations and simplifications of a given physical problem; those arising from the fact that certain parameters may not be known precisely; those arising from the approximate character of the numerical form given the mathematical problem, *i.e.*, truncation errors; and finally, those arising from the errors of the computing machine itself, *i.e.*, the "noise" or rounding errors. The main purpose of the present exposition is to treat this last category, and to discuss its great importance when large scale calculations are undertaken. Several examples are given to indicate generally how analyses of rounding errors can be performed.

3. *Minimal sets*, by Professor J. C. Oxtoby, Bryn Mawr College.

The theory of the structure of minimal sets due to Mariano Garcia, Jr., and G. A. Hedlund (*Bull. Amer. Math. Soc.*, vol. 54, (1948), pp. 954–964) was discussed, and it was shown that in the example there constructed of a 0-reducible minimal set which is not completely reducible the set of non-regularly almost periodic points actually consists of just two orbits, namely the orbit of the point β used in defining the set, and that of the point β' which is equal to β except for $n=0$.

4. *On smoothing operations*, by Professor I. J. Schoenberg, University of Pennsylvania.

C. O. OAKLEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, May 6, 1950.
ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, May 6, 1950.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Macalester College, St. Paul, May 6, 1950.
MISSOURI
NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.
NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of Washington, Seattle, June 16, 1950.
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